Perturbed Evolution Equations Governed by t- ω -Dissipative Operators

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1. Introduction.

In this paper we consider the perturbed evolution equation

$$(\mathrm{CP}: x_0)\quad \begin{cases} (d/dt)\,u(t) \in A(t)\,u(t) + f(t, u(t)) & \text{for } t \in (0, T), \\ u(0) = x_0 \end{cases}$$

where the operators $A(t)$ are t-ω-dissipative in a Banach space X, from the viewpoint of the difference approximation.

Crandall and Liggett [1], Kenmochi and Oharu [5], Kobayashi [6], and others studied homogeneous and time-independent equations ($A(t) = A$, $f(t, u) = 0$), Crandall and Pazy [2] studied homogeneous equations ($f(t, u) = 0$), Evans [3] studied equations where $A(t)$ are m-dissipative and $f(t, u) = f(t)$, and Pavel [7] and Iwamiya, Oharu and Takahashi [4] studied equations where $f(t, u) = 0$ or $f(t, u) = f(t)$.

We will prove the existence of a mild solution of the perturbed evolution equation (CP : x_0) for t-ω-dissipative operators $A(t)$.

2. Preliminaries.

Let X and X^* be a real Banach space and its dual space with norm $\|\cdot\|$. By an operator A in X we mean a mapping X into 2^X with domain $D(A) = \{x \in X ; Ax \neq \phi\}$ and range $R(A) = \cup\{Ax ; x \in D(A)\}$. As usually, we identify the operator A with its graph, so that we write $[x, y] \in A$ if $y \in Ax$. For each $x \in D(A)$, we write $|Ax| = \inf\{\|y\| ; y \in Ax\}$. We denote by $< x, f >$ the natural pairing between $x \in X$ and $f \in X^*$, and by F the duality mapping from X into X^*, i. e.,

(2.1) $\qquad F(x) = \{f \in X^* ; \ \|f\|^2 = \|x\|^2 = < x, f >\} \quad \text{for } x \in X.$

For $x, y \in X$, we define

(2.2) $\qquad < y, x >_i = \inf\{< y, f > ; f \in F(x)\}.$

It is known that there exists $x^* \in F(x)$ such that $< y, x^* > = < y, x >_i$.

Let ω be a real number. An operator A in X is said to be ω-dissipative if for any $[x_i, y_i] \in A$ ($i = 1, 2$),

$$< y_1 - y_2, x_1 - x_2 >_i \leqq \omega \|x_1 - x_2\|^2 .$$

A 0-dissipative operator is simply called dissipative. Apparently A is ω-dissipative if and only if $A - \omega I$ is dissipative.

A dissipative operator A is called m-dissipative if $R(I-\lambda A)=X$ for some $\lambda>0$. It is known that then $R(I-\lambda A)=X$ for any $\lambda>0$.

Now following Pavel [7], we introduce the following notion.

DEFINITION 2.1. *A family of operators* $\{A(t),\ t\in[0,T]\}$ *in X is said to be t-ω-dissipative if there exists a function* $L:[0,\infty)\to[0,\infty)$, *bounded on bounded set, and a real number* ω *such that*

(A 1) $\langle y_1-y_2,\ x_1-x_2\rangle_i\leq\omega\|x_1-x_2\|^2+(t_2-t_1)\|x_1-x_2\|L(\|x_1\|)$

for each $0\leq t_1\leq t_2\leq T$ and $[x_i,y_i]\in A(t_i)$ $(i=1,\ 2)$.

In following we set $D(t)=D(A(t))$ for $t\in[0,T]$.

And

(A 2) The multivalued mapping $t\to D(t)$ from R_+ to 2^X is said to be closed if : the conditions $t_n\to t$, $x_n\in D(t_n)$ with $x_n\to x$, imply $x\in\overline{D(t)}$.

Then we have

LEMMA 2.2. (see Pavel [7]) *If a family of operators* $\{A(t),\ t\in[0,T]\}$ *in X is t-ω-dissipative then we have*

$$(1-\lambda\omega)\|x-u\|\leq\|x-u-\lambda y\|+\lambda|A(s)u|+(t-s)L(\|u\|)$$

for any $[x,y]\in A(t)$, $u\in D(s)$, $0\leq s\leq t\leq T$ and $\lambda>0$.

3. Approximate solutions.

In this section we study the covergence of difference approximation of the Cauchy problem for the perturbed evolution equation (CP : x_0).

Throughout this paper, we set following hypothesis on $f(t,x)$, perturbed term of (CP : x_0)

(B) $f(t,x)$ is a function : $[0,T]\times X\to X$, satisfying : there exists a constant $M\ (>0)$ such that

$\|f(t,x)-f(s,y)\|\leq M(|t-s|+\|x-y\|)$

for s, $t\in[0,T]$ and $x,\ y\in X$.

Now we consider (CP : x_0) on finite interval $[0,T]$.

Let $\{u_n(t)\}$ be a sequence of X-valued simple function on $[0,T]$ defined by

$$u_n(t)=\begin{cases} x_0^n & \text{for } t=0 \\ x_i^n & \text{for } t\in[t^n_{i-1},t_i^n]\cap[0,T] \quad i=1,2,\cdots\cdots,N_n \end{cases} \tag{3.1}$$

and $n\geq1$ where $\{t_i^n\}$ represents the partition $\Delta_n=\{0=t_0^n<t_1^n<\cdots<t^n_{N_{n-1}}<T\leq t^n_{N_n}\}$ of the interval $[0,T]$ satisfying the condition

$$|\Delta_n|=\max_{1\leq i\leq N_n}(t_i^n-t^n_{i-1})\to0 \qquad \text{as } n\to\infty. \tag{3.2}$$

DEFINITION 3.1. (Kobayashi [6]) *Let* $x_0\in X$. *We say that the sequence* $\{u_n(t)\}$ *is a DS-approximate solution of* (*CP* : x_0) *if it satisfies*

$$\begin{cases} (x_i^n-x^n_{i-1})/(t_i^n-t^n_{i-1})-p_i^n\in A(t_i^n)x_i^n+f(t^n_{i-1},x^n_{i-1}) \\ i=1,2,\cdots\cdots N_n,\quad n\geq1,\quad x_0^n\to x_0 \quad \text{as } n\to\infty \end{cases} \tag{3.3}$$

and

(3.4) $$b_n = \sum_{i=1}^{N_n} \| p_i{}^n \| (t_i{}^n - t^n{}_{i-1}) \to 0 \quad \text{as } n \to \infty.$$

The following is the main result of this paper.

THEOREM 3.2. *Let* $T>0$ *and* $x_0 \in \overline{D(0)}$. *Assume* $(A1)$, $(A2)$ *and* (B). *Then following properties hold* :

(1) *There exists a continuous function* $u(t)$: $[0, T] \to X$, *such that any sequence* $\{u_n(t)\}$ *of DS-approximate solution of* $(CP : x_0)$ *is convergent to* $u(t)$ *as* $n \to \infty$, *uniformly on* $[0, T]$.

(2) $u(t) \in \overline{D(t)}$ *for* $t \in [0, T]$.

We say $u(t)$ in Theorem 3. 2 a mild solution of $(CP : x_0)$ on $[0, T]$.

4. Proof of Theorem 3.2.

The proof of Theorem 3. 2 is based on the following Lemmas.

We set $\omega_0 = \max(0, \omega)$ and, for simplicity, write $h_i = t_i{}^n - t^n{}_{i-1}$, $x_i = x_i{}^n$, $t_i = t_i{}^n$, $N = N_n$ etc.

Suppose that assumptions in Theorem 3.2 are satisfied.

LEMMA 4. 1. *Let* $u \in D(0)$, *then there exists a constant* $K_1 = K_1(x_0, u) > 0$ *such that* $\|x_i\| \leqq K_1$ *for all i and n where* $\omega_0 |\Delta| \leqq 1/2$.

PROOF. By Lemma 2.2,

$$(1 - \omega_0 h_i) \|x_i - u\| \leqq \|x_i - u - h_i y_i\| + h_i |A(0)u| + h_i t_i L(\|u\|)$$

$$\text{for any } [x_i, y_i] \in A(t_i).$$

From Definition 3. 1 there exists $y_i \in A(t_i) x_i$, such that

$$x_i - h_i y_i = x_{i-1} + h_i p_i + h_i f(t_{i-1}, x_{i-1}),$$

so

$$(1 - \omega_0 h_i) \|x_i - u\| \leqq \|x_{i-1} - u\| + h_i [\|p_i\| + \|f(t_{i-1}, x_{i-1})\| + |A(0)u| + t_i L(\|u\|)]$$

(4.1) $$\leqq (1 + Mh_i) \|x_{i-1} - u\| + h_i [\|p_i\| + M t_{i-1} + L(\|u\|) t_i] + h_i [\|f(0, u)\| + |A(0)u|]$$

where we used $\|f(t_{i-1}, x_{i-1})\| \leqq \|f(t_{i-1}, x_{i-1}) - f(0, u)\| + \|f(0, u)\| \leqq M(t_{i-1} + \|x_{i-1} - u\|) + \|f(0, u)\|$.

Inductively (4.1) implies

(4.2) $$\|x_i - u\| \leqq [\prod_{k=1}^{i} (1 - \omega_0 h_k)(1 + M h_k)] [\|x_0{}^n - u\| + \sum_{k=1}^{i} h_k \|p_k\| + \sum_{k=1}^{i} h_k t_{k-1} M + \sum_{k=1}^{i} h_k t_k L(\|u\|) + t_i (\|f(0, u)\| + |A(0)u|)]$$

Using the inequality

$$(1-t)^{-1} \leqq \exp(2t) \quad \text{for } t \in [0, 1/2] \quad \text{and} \quad 1 + t \leqq \exp(t),$$

we have

(4.3) $$\|x_i - u\| \leqq \exp((2\omega_0 + M)T)\,[\|x_0^n - u\| + b_n + (M + L(\|u\|))T^2 + (\|f(0,u)\| + |A(0)u|)T]$$

Since $\|x_0^n - u\| \to \|x_0 - u\|$ and $b_n \to 0$ as $n \to \infty$ by Definition 3.1, then there exists a constant $K_1 = K_1(u, x_0) > 0$ such that

(4.4) $$\|x_i\| \leqq K_1 \quad \text{for } n = 1, 2, \cdots\cdots \text{ and } i = 1, 2, \cdots\cdots N.$$

Let $\{\hat{u}_m\}$ be another DS-approximate solution of (CP : x_0). We use the methode of Kobayashi [6].

LEMMA 4.2. *Let* $a_{i,j} = \|x_i - \hat{x}_j\|$. *Then,*

(4.5) $$(h_i + \hat{h}_j)a_{i,j} \leqq \exp(2(M+\omega_0)\min(h_i, \hat{h}_j))\,[\hat{h}_j \exp(Mh_i)a_{i-1,j} + h_i \exp(M\hat{h}_j)a_{i,j-1} + h_i\hat{h}_j(\|p_i\| + \|\hat{p}_j\| + M|t_{i-1} - \hat{t}_{j-1}| + K|t_i - \hat{t}_j|)]$$

for $0 \leqq i \leqq N$ *and* $0 \leqq j \leqq \hat{N}$ *with* $(M+\omega_0)|\varDelta|, (M+\omega_0)|\hat{\varDelta}| \leqq 1/2$.
Here $K = \max\{L(\|x_i\|), L(\|\hat{x}_j\|) : 0 \leqq i \leqq N_n, 0 \leqq j \leqq \hat{N}_m, n, m \geqq 1\}$ *and the notations with symbol "^" correspond to the solution* $\hat{u}_m(t)$.

PROOF. By Definition 3.1, there exists $y_i \in A(t_i)x_i$ and $\hat{y}_j \in A(\hat{t}_j)\hat{x}_j$ such that
$$x_i - h_i y_i = x_{i-1} + h_i(p_i + f(t_{i-1}, x_{i-1})),$$
$$\hat{x}_j - \hat{h}_j\hat{y}_j = \hat{x}_{j-1} + \hat{h}_j(\hat{p}_j + f(\hat{t}_{j-1}, \hat{x}_{j-1})),$$
and there exists $x^* \in F(x_i - \hat{x}_j)$ such that
$$\langle y_i - \hat{y}_j, x^* \rangle = \langle y_i - \hat{y}_j, x_i - \hat{x}_j \rangle_i .$$
Then

(4.6) $$\begin{aligned}(h_i + \hat{h}_j)a_{i,j} &= \hat{h}_j\langle x_i - \hat{x}_j, x^*\rangle + h_i\langle \hat{x}_j - x_i, -x^*\rangle \\ &= \hat{h}_j\langle x_i - h_i y_i - \hat{x}_j, x^*\rangle + h_i\langle \hat{x}_j - \hat{h}_j\hat{y}_j - x_i, -x^*\rangle + h_i\hat{h}_j\langle y_i - \hat{y}_j, x^*\rangle \\ &= \hat{h}_j\langle x_{i-1} - \hat{x}_j, x^*\rangle + h_i\langle \hat{x}_{j-1} - x_i, -x^*\rangle \\ &\quad + h_i\hat{h}_j[\langle y_i - \hat{y}_j, x^*\rangle + \langle p_i - \hat{p}_j, x^*\rangle + \langle f(t_{i-1}, x_{i-1}) - f(\hat{t}_{j-1}, \hat{x}_{j-1}), x^*\rangle].\end{aligned}$$

By (A1), (B) and Lemma 4.1, we have

(4.7) $$(h_i + \hat{h}_j)a_{i,j} \leqq \hat{h}_j a_{i-1,j} + h_i a_{i,j-1} + h_i\hat{h}_j[\omega_0 a_{i,j} + K|t_i - \hat{t}_j| + \|p_i\| + \|\hat{p}_j\| + M(|t_{i-1} - \hat{t}_{j-1}| + a_{i-1,j-1})].$$

Here $a_{i-1,j-1} \leqq a_{i-1,j} + a_{i,j-1} + a_{i,j}$, then (4.7) implies

(4.8) $$(h_i + \hat{h}_j)(1 - (M+\omega_0)h_i\hat{h}_j/(h_i + \hat{h}_j))\,a_{i,j} \leqq \hat{h}_j(1 + Mh_i)a_{i-1,j} + h_i(1 + M\hat{h}_j)a_{i,j-1} + h_i\hat{h}_j[\|p_i\| + \|\hat{p}_j\| + M|t_{i-1} - \hat{t}_{j-1}| + K|t_i - \hat{t}_j|].$$

Noted $1 + Mh_i \leqq \exp(Mh_i)$, $1 + M\hat{h}_j \leqq \exp(M\hat{h}_j)$ and $[1 - (M+\omega_0)h_i\hat{h}_j/(h_i + \hat{h}_j)]^{-1} \leqq \exp(2(M+\omega_0)\min(h_i, \hat{h}_j))$ for $(M+\omega_0)|\varDelta| \leqq 1/2$, $(M+\omega_0)|\hat{\varDelta}| \leqq 1/2$, Lemma 4.2 is proved.

LEMMA 4.3. *Let* $u \in D(0)$. *Then*

(4.9) $$a_{i,j} \leqq \exp(3M_1(t_i+\hat{t}_j))[\|x_0{}^n-u\|+\|\hat{x}_0{}^m-u\|+\{(t_i-\hat{t}_j)^2+(t_i+\hat{t}_j)\times(|\Delta|+|\Delta|)\}^{\frac{1}{2}}(K_3+2Mt_i+2K\hat{t}_j)+\sum_{k=1}^{i} h_k\|p_k\|+\sum_{k=1}^{j}\hat{h}_k\|\hat{p}_k\|]$$

for $0 \leqq i \leqq N$ and $0 \leqq j \leqq \hat{N}$ with $N|\Delta|,\ \hat{N}|\hat{\Delta}| \leqq 1/2$

where $M_1 = M+\omega_0$ and $K_3 = \|f(0,u)\|+|A(0)u|+(M+L(\|u\|))T$.

PROOF. By (4.2) in the proof of Lemma 4.1, we have

$$a_{i,0} = \|x_i-\hat{x}_0{}^m\| \leqq \|x_i-u\|+\|\hat{x}_0{}^m-u\|$$
$$\leqq \exp(2M_1t_i)[\|x_0{}^n-u\|+\sum_{k-1}^{i} h_k\|p_k\|+t_iK_3]+\|\hat{x}_0{}^m-u\|.$$

Then $a_{i,0}$ satisfies (4.9) for $0 \leqq i \leqq N$. Similarly $a_{0,j}$ satisfies (4.9) for $0 \leqq j \leqq \hat{N}$. Now assume that $a_{i-1,j}$ and $a_{i,j-1}$ satisfy (4.9) for induction. By (4.5) in Lemma 4.2, we have

(4.10) $$\begin{aligned}(h_i+\hat{h}_j)a_{ij} &\leqq \exp(2M_1\min(h_i,\hat{h}_j)[\hat{h}_j\exp(Mh_i)a_{i-1,j}+h_i\exp(M\hat{h}_i)a_{i,j-1})\\ &\quad +h_i\hat{h}_j(\|p_i\|+\|\hat{p}_j\|+M|t_{i-1}-\hat{t}_{j-1}|+K|t_i-\hat{t}_j|)]\\ &\leqq \exp(3M_1(t_i+\hat{t}_j)H.\end{aligned}$$

Here

(4.11) $$\begin{aligned}H &= (h_i+\hat{h}_j)(\|x_0{}^n-u\|+\|\hat{x}_0{}^m-u\|)\\ &\quad +K_3(\hat{h}_jc_{i-1,j}+h_ic_{i,j-1})\\ &\quad +2M(\hat{h}_jt_{i-1}c_{i-1,j}+h_it_ic_{i,j-1})\\ &\quad +2K(\hat{h}_j\hat{t}_jc_{i-1,j}+h_i\hat{t}_{j-1}c_{i,j-1})\\ &\quad +[\hat{h}_j(\sum_{k=1}^{i-1}h_k\|p_k\|+\sum_{k=1}^{j}\hat{h}_k\|\hat{p}_k\|)+h_i(\sum_{k=1}^{i}h_k\|p_k\|+\sum_{k=1}^{j-1}\hat{h}_k\|\hat{p}_k\|)\\ &\qquad +h_i\hat{h}_j(\|p_i\|+\|\hat{p}_j\|)]\\ &\quad +h_i\hat{h}_jM|t_{i-1}-\hat{t}_{j-1}|\\ &\quad +h_i\hat{h}_jK|t_i-\hat{t}_j|\end{aligned}$$

where $c_{i,j} = \{(t_i-\hat{t}_j)^2+(t_i+\hat{t}_j)(|\Delta|+|\hat{\Delta}|)\}^{\frac{1}{2}}$

We denote by P_q the q-th term ($q = 1, 2 \cdots 7$) of the right hand side of equality (4.11). Using Cauchy-Schwartz's inequality (see Kobayashi [6]), we have

(4.12) $$P_2 \leqq K_3(h_i+\hat{h}_j)c_{i,j}.$$

Next $$P_3 = 2Mt_i(\hat{h}_jc_{i-1,j}+h_ic_{i,j-1})-2M\hat{h}_ih_jc_{i-1,j}.$$

Here $$\begin{aligned}2Mh_i\hat{h}_jc_{i-1,j} &\geqq Mh_i\hat{h}_j|t_{i-1}-\hat{t}_j|+Mh_i\hat{h}_j\hat{t}_j{}^{\frac{1}{2}}|\hat{\Delta}|^{\frac{1}{2}}\\ &\geqq Mh_i\hat{h}_j|t_{i-1}-\hat{t}_j|+Mh_i\hat{h}_j{}^2\\ &\geqq Mh_i\hat{h}_j(|t_{i-1}-\hat{t}_j|+\hat{h}_j)\\ &\geqq P_6,\end{aligned}$$

therefore $P_3+P_6 \leqq 2Mt_i(\hat{h}_jc_{i-1,j}+h_ic_{i,j-1})$.

By the similar calculation to P_2, we have

(4.13) $P_3+P_6 \leqq 2Mt_i(h_i+\hat{h}_j)c_{i,j}$.

Similarly we have

(4.14) $P_4+P_7 \leqq 2K\hat{t}_j(h_i+\hat{h}_j)c_{i,j}$.

Last, apparently

(4.15) $P_5 \leqq (h_i+\hat{h}_j)(\sum_{k=1}^{i} h_k \|p_k\| + \sum_{k=1}^{j} \hat{h}_k \|\hat{p}_k\|)$.

Rearranging (4.11)−(4.15), Lemma 4.3 is proved.

Proof of Theorem 3.2. Let $\{u_n(t)\}$ be a DS-approximate solution of $(CP : x_0)$ with $x_0 \in \overline{D(0)}$. Let $\{u_p\} \subset D(0)$ be a sequence such that $u_p \to x_0$ as $p \to \infty$. Then by Lemma 4.3, we have

(4.16) $$\|x_i{}^n - x_j{}^m\| \leqq \exp(3M_1(t_i{}^n+t_j{}^m))\,[\,\|x_0{}^n-u_p\| + \|x_0{}^m-u_p\| + \{(t_i{}^n-t_j{}^m)^2+(t_i{}^n+t_j{}^m)(|\Delta_n|+|\Delta_m|)\}^{\frac{1}{2}}(K_3+2Mt_i{}^n+2Kt_j{}^m)+b_n+b_m]$$

for $0 \leqq i \leqq N_n$, $0 \leqq j \leqq N_m$, $p \geqq 1$ and $n, m \geqq 1$ such that $N_n|\Delta_n|$, $N_m|\Delta_m| \leqq 1/2$. Lettng $n, m \to \infty$ with $t_i{}^n, t_j{}^m \to t$ in (4.16), we have

$$\limsup_{t_i{}^n, t_j{}^m \to t} \|x_i{}^n - x_j{}^m\| \leqq 2\exp(6M_1 t)\|x_0-u_p\| \quad \text{for } p \geqq 1.$$

Since $u_p \to x_0$ as $p \to \infty$, we see that there exists

(4.17) $$u(t) = \lim x_i{}^n \quad \text{as } t_i{}^n \to t,\ n \to \infty,$$
$$= \lim_{n\to\infty} u_n(t) \quad \text{uniformly for } t \in [0, T].$$

Furthermore letting $t_i{}^n \to t$, $t_j{}^m \to s$, $n, m \to \infty$ in (4.16), we have

(4.18) $$\|u(t)-u(s)\| \leqq \exp(3M_1(t+s))\,[2\|x_0-u_p\| + |s-t|(K_3+2Mt+2Ks)]$$

for $t, s \in [0, T]$ and $p \geqq 1$. This shows that $u(t)$ is continuous on $[0, T]$. At last, by assumption (A2) and (4.17), $u(t) \in \overline{D(t)}$.

5. Existence of DS-approximate solutions.

In this section, we shall give a sufficient condition to have DS-approximate solution of $(CP : x_0)$.

We consider following conditions on $\{A(t)\}$ and $f(t, x)$:

(C) $\liminf_{h \to 0} h^{-1} d[R(I-h(A(t+h)+f(t,x))), x] = 0$ for any $t \in [0, T]$ and $x \in \overline{D(t)}$,

(A3) For each $t \in (0, \infty)$, $[x, y] \in A(t)$ and $t_n \uparrow t$ as $n \to \infty$, there exists $[x_n, y_n] \in A(t_n)$ such that $[x_n, y_n] \to [x, y]$ as $n \to \infty$.

Then we have

THEOREM 5.1. *Assume that conditions (A1) (A2) (A3) (B) and (C) are satisfied. Then for each* $T > 0$, $x_0 \in \overline{D(0)}$ *and n (a natural number) there exists a DS-approximate solution*

$u_n(t)$ *with* $(M+\omega_0)\times|\Delta_n|\leqq 1/2,\ |\Delta_n|\leqq 1/n,$ *and* $\|p_i^n\|\leqq 1/n,\ i=1,2\cdots\cdots N.$

PROOF. We mimic the proof of Pavel [7 , Thorem 5.1] (and so Kobayashi [6 , Theorem 5.1] .

Let $x_0\in\overline{D(0)}$ and n such that $2\omega_0/n\leqq 1$. Then for each $x\in\overline{D(t)}$, define $\delta(x,y)$ as follows

(5.1) $$\delta^n(x,y)=\sup\{h;0<h\leqq 1/n, ^\exists[x_h,y_h]\in A(t+h):\|x_h-h(y_h+f(t,x))-x\|<h/n\}.$$

By (C), $\delta^n(x,t)>0$ for any $t\geqq 0$ and $x\in\overline{D(t)}$. Choose $x_0{}^n\in D(0)$ such that $x_0{}^n\to x_0$ as $n\to\infty$. Inductively, choose $h_i{}^n\in[\delta^n(x^n{}_{i-1},t^n{}_{i-1})/2,1/n]$, $i=1,2\cdots$, and $[x_i{}^n,t_i{}^n]\in A(t_{i-1}{}^n+h_i{}^n)$ with the properties

(5.2) $$\|x_i{}^n-h_i{}^n(y_i{}^n+f(t^n{}_{i-1},x^n{}_{i-1}))-x^n{}_{i-1}\|<h_i{}^n/n,\ x^n{}_{i-1}\in D(t^n{}_{i-1}),t_0{}^n=0$$

Set

(5.3) $$p_i{}^n=(h_i{}^n)^{-1}[x_i{}^n-x^n{}_{i-1}-h_i{}^n(y_i{}^n+f(t^n{}_{i-1},x^n{}_{i-1}))],\quad t^n{}_i=t^n{}_{i-1}+h_i{}^n.$$

Then $\|p_i{}^n\|\leqq 1/n$ and (3.2) (3.3) are satisfied. We will prove

$\lim_{i\to\infty} t_i{}^n=\sum_{k=1}^{\infty}h_k{}^n=+\infty$, so (3.4) holds for any $T>0$. Assume by contradiction that $\lim_{i\to\infty} t_i{}^n=\bar{t}<\infty$.

We set $a_{i,j}=\|x_i-x_j\|$ where write, for simplicity, $x_i=x_i{}^n$, $h_i=h_i{}^n$, $t_i=t_i{}^n$ etc. From (4.2) in the proof of Lemma 4.1 with $u=x_0{}^n$, we have

(5.4) $$a_{i,0}\leqq\exp(2\omega_0+M)t_i)\Big[\sum_{k=1}^{i}h_k\|p_k\|+t_i(\|f(0,x_0{}^n)\|+|A(0)x_0{}^n|)+\sum_{k=1}^{i}h_kt_{k-1}M+\sum_{k=1}^{i}h_kt_kL(\|x_0{}^n\|)\Big].$$

Similar to (4.4) in Lemma 4.1, we prove that there exists a constant $K_1=K_1(x_0{}^n,\bar{t})>0$, such that $\|x_i\|\leqq K_1$ for all $i=1,2\cdots\cdots$.

Therefore, there exists $K=K(x_0{}^n,\bar{t})>0$ such that $L(\|x_i\|)\leqq K$ for $i=0,1\cdots\cdots$.

Now we claim that for $i\geqq j\geqq k\geqq 1$,

(5.5) $$a_{i,j}\leqq\exp(3(\omega_0+M)(t_i+t_j-2t_k))\Big[(t_i-t_j)(|A(t_k)x_k|+\|f(t_k,x_k)\|)+\sum_{s=k+1}^{i}h_s\|p_s\|+\sum_{s=k+1}^{j}h_s\|p_s\|+M\sum_{s=j+1}^{i}h_s(t_{s-1}-t_k)+K\sum_{s=j+1}^{i}h_s(t_s-t_k)\Big].$$

Similar to (5.4), we have

$$a_{i,k}\leqq\exp((2\omega_0+M)(t_i-t_k))\Big[\sum_{s=k+1}^{i}h_s\|p_s\|+(t_i-t_k)(|A(t_k)x_k|+\|f(t_k,x_k)\|)+M\sum_{s=k+1}^{i}h_s(t_{s-1}-t_k)+K\sum_{s=k+1}^{i}h_s(t_s-t_k)\Big],$$

then $a_{i,k}$ satisfies (5.5) and clearly $a_{i,i}$ also satisfies (5.5).

Assume for induction that $a_{i-1,j}$ and $a_{i,j-1}$ satisfy (5.5) with $i > j > k$. Similar to Lemma 4.2, we can prove

$$(5.6) \qquad (h_i + h_j)\, a_{i,j} \leqq \exp(2(M+\omega_0)\min(h_i, h_j))\,[h_j \exp(M h_i)\, a_{i-1,j} + h_j \exp(M h_j)\, a_{i,j-1} + h_i h_j(\|p_i\| + \|p_j\| + M(t_{i-1} - t_{j-1}) + K(t_i - t_j))].$$

Hence, by the inductive assumption, (5.6) implies

$$(5.7) \qquad (h_i + h_j)\, a_{i,j} \leqq \exp(3(M+\omega_0)(t_i + t_j - 2t_k))\, H$$

where
$$\begin{aligned} H = {} & [h_j(t_{i-1} - t_j) + h_i(t_i - t_{j-1})](|A(t_k)x_k| + \|f(t_k, x_k)\|) \\ & + [h_j(\sum_{s=k+1}^{i-1} h_s\|p_s\| + \sum_{s=k+1}^{j} h_s\|p_s\|) + h_i(\sum_{s=k+1}^{i} h_s\|p_s\| + \sum_{s=k+1}^{j-1} h_s\|p_s\|) \\ & \qquad + h_i h_j(\|p_i\| + \|p_j\|)] \\ & + M[h_j \sum_{s=j+1}^{i-1} h_s(t_{s-1} - t_k) + h_i \sum_{s=j}^{i} h_s(t_{s-1} - t_k) + h_i h_j(t_{i-1} - t_{j-1})] \\ & + M[h_j \sum_{s=j+1}^{i-1} h_s(t_s - t_k) + h_i \sum_{s=j}^{i} h_s(t_s - t_k) + h_i h_j(t_i - t_j)]. \end{aligned}$$

By easy calculatins, we have

$$\begin{aligned} H = {} & (h_i + h_j)(t_i - t_j)(|A(t_k)x_k| + \|f(t_k, x_k)\|) \\ & + (h_i + h_j)(\sum_{s=k+1}^{i} h_s\|p_s\| + \sum_{s=k+1}^{j} h_s\|p_s\|) \\ & + (h_i + h_j) M \sum_{s=j+1}^{i} h_s(t_{s-1} - t_k) \\ & + (h_i + h_j) K \sum_{s=j+1}^{i} h_s(t_s - t_k) \end{aligned}$$

Hence, (5.5) is proved. By $\|p_i\| \leqq 1/n$, $t_i \leqq \bar{t}$, it follows

$$\sum_{s=k+1}^{i} h_s\|p_s\| \leqq (t_i - t_k)/n \leqq \bar{t} - t_k, \quad M\sum_{s=k+1}^{i} h_s(t_{s-1} - t_k) \leqq M(\bar{t} - t_k)^2 \quad \text{a.s.o.}$$

Then from (5.5), we have

$$(5.8) \qquad \limsup_{i,j\to\infty} \|x_i - x_j\| \leqq \exp(6(\omega_0 + M)(\bar{t} - t_k))(\bar{t} - t_k)[2 + (M+K)(\bar{t} - t_k)] \quad \text{for } k \geqq 1.$$

As $k \to \infty$, $t_k \to \bar{t}$, then we see that $\lim_{i\to\infty} x_i = \bar{x}$ exists. Since $x_i \in D(t_i)$, by (A2), it follows $\bar{x} \in \overline{D(\bar{t})}$. By (C), there exists $h^* \in (0, 1/n)$ and $[x_{h^*}, y_{h^*}] \in A(\bar{t} + h^*)$, such that

$$(5.9) \qquad \|x_{h^*} - h^*(y_{h^*} + f(\bar{t}, \bar{x})) - \bar{x}\| < h^*/2n.$$

On the other hand, $h_i \to 0$ as $i \to \infty$, therefore $\delta^n(t_i, x_i) \to 0$ as $i \to \infty$, then $\delta^n(t_i, x_i) < h^*$ for all $i \geqq i_0(h^*)$.

By the definition of $\delta^n(t_i, x_i)$, it follows

$$(5.10) \qquad \|\tilde{x}_i - h^*(\tilde{y}_i + f(t_i, x_i)) - x_i\| \geqq h^*/n \quad \text{for all } [\tilde{x}_i, \tilde{y}_i] \in A(t_i + h^*),\ i \geqq i_0(h^*).$$

Since $t_i + h^* \uparrow \bar{t} + h^*$ as $i \to \infty$, by (A3) there exists $[\tilde{x}_i, \tilde{y}_i] \in A(t_i + h^*)$ such that $\tilde{x}_i \to x_{h^*}$, $\tilde{y}_i \to y_{h^*}$ as $i \to \infty$. Letting $i \to \infty$ in (5.10) we have

$\| x_{h^*} - h^* (y_{h^*} + f(\bar{t}, \bar{x})) - \bar{x} \| \geq h^*/n$ which contradicts (5.9).

Thus we proved that $t_i \to \infty$ as $i \to \infty$. Hence Theorem 5.1 is proved.

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