New Three-Dimension Problems in Queueing System Analysis

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Abstract

This monograph discusses a new type of three-class priority queues with controllable parameters $(k_1, k_2, k_3 \geq 1)$, which operates as follows; a single server continues serving messages in queue n until the number of messages (customers) decreases to k_n less than that found upon the server's last arrival at queue $n, n = 1, 2, 3$. In succession, messages of the highest class present in the system, if any, will be served according to this k_n-decreasing service. Laplace-Stieltjes transform (LST) for the waiting time distribution of each class messages are explicitly obtained by using the generating function (GF) method for $M_1, M_2, M_3/G_1, G_2, G_3/1$ priority queueing system with the multiple vacation (MV) rule. The generalization of dimension-number and the decomposition law (property) are also considered for the special cases.

Contents:

Keywords: general decrementing service; functional equation; probability generating function;

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1 Introduction

Classical paper for priority queues may be known by H. Kesten and J. Th. Runnenbeurg (1957) in [1]. There have been some analytical studies of priority queueing models with controllable parameters. A flexible queueing model with a general decreasing service priority scheme is proposed by Katayama (1992). The priority descipline is defined by a vector $(k_1, k_2, \cdots, k_N)$ in [7], where $1 \leq k_n \leq \infty, n = 1, 2, \cdots, N$ and operates as follows: For the moment, suppose that class-n messages are served. A single server continues serving messages in queue n until either queue n becomes empty, or the number of messages decreases to k_n less than that found upon the server's last arrival at queue n, whichever occurs first. This service discipline for messages in queue n is called a general (decrementing) decreasing service, the k_n-decreasing service or the k_n-busy period service. If there are messages in the system at the completion epoch of the k_n-decreasing service for the class-n messages, messages of the highest class present in the system, say class i, will be served next according to the k_i-decreasing service. If there is no message in the system, the server goes away for a vacation. The vacation may be repeated if the server finds the system still empty at the end of a vacation. If any, the server starts serving messages of the highest class present in the system according to the k_n decreasing service. This is called the multiple vacation (MV) rule. Takagi [3] derived the mean message waiting time formula in a symmetric cyclic-service polling system with $(k_1 = k_2 = \cdots = k_N = 1)$. By using the boundary value method, Cohen [5] analyzed a semi-exhaustive alternating service queueing model $(N = 2)$ with $(k_1 = k_2 = 1)$. If $k_1 = k_2 = \cdots = k_n = \infty$, then the priority discipline reduces to the exhaustive service priority discipline. In the case of $(k_1 = k_2 = \infty)$, it also reduces to the alternating priority discipline.

Famous historical two-dimension $(N = 2)$ problem in queueing analysis may be the shortest queue problem, for which Fayolle and Iasnogorodski [2] have analyzed by using the boundary value technique for the first time (1979). The shortest queue problem is one of the lane selection problems. With using the boundary value technique, similar two-queue models $(N = 2)$ have been analyzed by some researchers, e.g. Cohen analyzed two queue, one-server model with priority for the longer queue, which is considered as a dual queueing system against the shortest queue model. (In Cohen's paper [6], the following is cited: Dr. T. Katayama has formulated the priority model with the longer queue and brought it to the author's attention during his visit in October/November 1984 to the NTT Electr. Communi. Lab.'s Musashino, Tokyo 180). However, the three-dimension $(N = 3)$ problem of the shortest queue has still been an open problem. Coffman, Fayolle and Mitrani [4] analyzed a flexible alternating service discipline with time-limited service (T_1, T_2) by using the boundary value method, where $T_n, n = 1, 2$ are called the maximum attendance time (MAT). Such a flexible priority discipline with controllable parameters is effective for the performance optimization and has a potential applicability to message processing with multi-class tasks and routing schems in communication systems.

In this monograph, we consider priority queues with the MV rule and controllable parameters $(k_1, k_2, \cdots, k_N)$, called k_n-decreasing service, which is one of the flexible service disciplines. Despite the effectiveness of flexible service disciplines, few analytical results in the literature have been obtained for flexible priority systems with multiple $(N \geq 3)$ message classes. This may be due to the difficulty and the complexity of queueing system analysis.

In the following, a single server services class-n messges at the service counter S_n. Class-n messages with arrival rate λ_n arrive to the infinite capacity queue Q_n. The Laplace-Stieltjes transform (LST) of the distribution function (DF), the mean, and the m-th moment of the service time H_n for each message of class-n are denoted by $H_n^*(s), h_n$ and $h_n^{(m)} (m = 2, 3, \cdots)$, respectively. Similarly, the LST of the DF, the mean, and the m-th moment of the vacation time V are denoted by $V^*(s), v$ and $v^{(m)} (m = 2, 3, \cdots)$, respectively. The switch-over time needed to switch service from one class to another is assumed to be zero. Messages within a class are served in each queue on the first-in-first-out (FIFO) discipline. The server utilization for class-n messages is denoted by $\rho_n := \lambda_n h_n$, where λ_n is Poisson arrival rate of class-n messages. The total server utilization is denoted by $\rho := \sum_{n=1}^{N} \rho_n < 1$, which guarantees the system stability.

2 Two-class Priority Queues ($N = 2$)

In this section, the results of Katayama [7] for a system with two-class priority queues with $(k_1, k_2 \geq 1)$ are summarized. Foremost, note that $(k_1, k_2 \geq 1)$ is equivalent to $(k_1 = \infty, k_2 \geq 1)$, since class-1 messages have priority over class-2 messages. We introduce some notations. The queueing system consists of two-parallel queues Q_1 and Q_2 for messages of class-1 and class-2, respectively. Messages in Q_1 and Q_2 are served by a single server in accordance with the priority discipline $(k_1 = \infty, k_2 \geq 1)$. After switching over to S_1, the server serves all messages in Q_1 until it becomes empty, i.e. $k_1 = \infty$. Just after service completion at S_1, the server switches over to S_2 and continues serving messages in Q_2 until either it becomes empty, or the number of messages decreases to k_2 less than that found upon its arrival at S_2, whichever occurs first, i.e. the k_2-decreasing service.

We use the following notation:

$$\begin{aligned} &r_n := \lambda_n/\lambda,\ n = 1, 2, && \lambda := \lambda_1 + \lambda_2, \\ &\rho_n := \lambda_n h_n,\ n = 1, 2, && \rho := \rho_1 + \rho_2, \\ &h := r_1 h_1 + r_2 h_2, && h^{(2)} := r_1 h_1^{(2)} + r_2 h_2^{(2)}, \end{aligned} \tag{1a}$$

and

$$\begin{aligned} &Q_n(x) := H_n^*\{\lambda_n(1-x)\}, \quad n = 1, 2, \\ &Q_v(x, y) := V^*\{\lambda_1(1-x) + \lambda_2(1-y)\} = V(x, y). \end{aligned} \tag{1b}$$

From the busy-period analyis on the standard $M/G/1$ queue, we then have LST $\Theta_n^*(s)$ for class-n messages:

$$\begin{aligned} &\Theta_n^*(s) = H_n^*\{s + \lambda_n(1 - \Theta_n^*(s))\}, \quad n = 1, 2, \\ &E(\Theta_n) = \frac{h_n}{1-\rho_n}, \quad E(\Theta_n^2) = \frac{h_n^{(2)}}{(1-\rho_n)^3}, \end{aligned} \tag{1c}$$

$$\begin{aligned} &G_1(y) := \Theta_1^*\{\lambda_2(1-y)\}, \\ &G_2(x) := \Theta_2^*\{\lambda_1(1-x)\}, \\ &G_n(x; m) := \{G_n(x)\}^m, \quad m = 1, 2, 3, \cdots. \end{aligned} \tag{1d}$$

For simplicity, we also define as, $Q_{3(mod2)} := Q_1,\ \lambda_3 := \lambda_1,\ h_3^{(2)} := h_1^{(2)},\ \rho_3 := \rho_1$.

2.1 Queue-Length GF (Generating Function) at Service-Completion Epochs

Firstly, we introduce:

$\phi_n(i, j)$: the steady-state probability that i messages are waiting in Q_1 and j messages are waiting in Q_2 just after service-completion epochs at $S_n, n = 1, 2, \quad i, j = 0, 1, 2, \cdots$, that is, when the server has completed the exhaustive service at $S_1(k_1 = \infty)$ or the $k_2(\geq 1)$-decreasing service at S_2 and

$\Phi_n(x, y) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi_n(i, j) x^i y^j \qquad |x| \leq 1,\ |y| \leq 1, \quad n = 1, 2.$

We have the following functional relations, where $\Phi(0, y) := \sum_{n=1}^{2} \Phi_n(0, y), \phi_0 := \Phi(0, 0)$:

$$\Phi_1(0, y) = \phi_0\{V(G_1(y), y) - V(0, y)\} + \Phi_2(G_1(y), y) - \Phi_2(0, y), \tag{2a}$$

$$\begin{aligned} \Phi_2(x, y) = &\{\Phi(0, y) - \Phi(0, 0)\} \frac{G_2(x)^{k_2}}{y^{k_2}} + \sum_{j=1}^{k_2-1} \phi(0, j) y^j \left\{ \frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}} \right\} \\ &+ \phi_0\{V(0, y) - V(0, 0)\} \frac{G_2(x)^{k_2}}{y^{k_2}} + \phi_0 \sum_{j=1}^{k_2-1} v(0, j) y^j \left\{ \frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}} \right\}. \end{aligned} \tag{2b}$$

Note that $V(0,0)=0$, because of multiple vacation rule. Eliminating $\Phi_2\{G_1(y),y\}$ from (2a) and (2b) after setting $x=G_1(y)$ in (2b), we get

$$\Lambda_1(y)\Phi(0,y)+\phi_0\frac{G_2(G_1)^{k_2}}{y^{k_2}}-\sum_{j=1}^{k_2-1}\phi(0,j)y^j\left\{\frac{G_2(G_1)^j}{y^j}-\frac{G_2(G_1)^{k_2}}{y^{k_2}}\right\}$$
$$=\phi_0V(G_1,y)-\phi_0\Lambda_1(y)V(0,y)+\phi_0\sum_{j=1}^{k_2-1}v(0,j)y^j\left\{\frac{G_2(G_1)^j}{y^j}-\frac{G_2(G_1)^{k_2}}{y^{k_2}}\right\}, \tag{3a}$$

where $\Lambda_1(y):=1-G_2\{G_1(y)\}^{k_2}/y^{k_2}$ and $G_1\equiv G_1(y)$. This is our basic functional equation with unknown function $\Phi(0,y)$ and unknown probabilities $\phi(0,j), j=1,2,\cdots,k_2-1$ and ϕ_0 for deriving of $W_1^*(s)$ and $W_2^*(s)$ later, which is equivalent to the following functional equation, where $\Lambda_0(y):=y^{k_2}-G_2\{G_1(y)\}^{k_2}$:

$$\Lambda_0(y)\Phi(0,y)-\sum_{j=1}^{k_2-1}\phi(0,j)\left\{y^{k_2}G_2(G_1)^j-y^jG_2(G_1)^{k_2}\right\}$$
$$=\phi_0\{V(G_1,y)y^{k_2}-G_2(G_1)^{k_2}-\Lambda_0(y)V(0,y)\} \tag{3b}$$
$$+\phi_0\sum_{j=1}^{k_2-1}v(0,j)\left\{y^{k_2}G_2(G_1)^j-y^jG_2(G_1)^{k_2}\right\}.$$

We know that $\Lambda_0(y)=0$ has the zeros, $y=\omega_r,\ r=1,2,\cdots,k_2-1$, see Lemma 2.1. Using the zeros, we obtain the following simultaneous linear equation for $\{\phi(0,j)\}$, for which the existence and the uniqueness are guaranteed by $\rho<1$:

$$\sum_{j=1}^{k_2-1}\alpha_j(\omega_r)\phi(0,j)=\phi_0\alpha_0(\omega_r),\quad r=1,2,\cdots,k_2-1, \tag{4}$$
$$\alpha_j(\omega_r):=\left\{y^jG_2(G_1)^{k_2}-y^{k_2}G_2(G_1)^j\right\}_{y=\omega_r}, \tag{4a}$$
$$\alpha_0(\omega_r):=\{V(G_1,y)^{k_2}-G_2(G_1)^{k_2}\}_{y=\omega_r}$$
$$+\sum_{j=1}^{k_2-1}v(0,j)\left\{y^{k_2}G_2(G_1)^j-y^jG_2(G_1)^{k_2}\}\right\}_{y=\omega_r}. \tag{4b}$$

Here, note that $\alpha_j(\omega_r)$ and $\alpha_0(\omega_r)$ are known functions.

Lemma 2.1. Let us consider a function for y,

$$\Lambda(y)=y^{k_2}-\hat{H}^*\{\lambda_2(1-y)\},\ \text{where}\ \hat{H}^*(s):=\Theta_2^*[\lambda_1\{1-\Theta_1^*(s)\}]^{k_2}. \tag{5}$$

Applying Takács lemma [14] to $\Lambda(y)=0$, we obtain the following results: If $\rho\le 1$, then the trancendental equation has exactly k_2 roots $\omega_r, r=1,2,\cdots,k_2-1$ in the unit circle $|y|<1$, while $\omega_{k_2}=1$, and $\omega_r, r=1,2,\cdots,k_2-1$ are explicitly expressed by

$$\omega_r:=\sum_{j=1}^{\infty}\frac{(-\lambda_2)^{j-1}\theta_r^j}{j!}\frac{d^{j-1}}{d\lambda_2^{j-1}}\left\{\hat{H}^*(\lambda_2)\right\}^{j/k_2}, \tag{5a}$$
$$\theta_r:=exp\left\{\frac{2\pi ri}{k_2}\right\},\quad i:=\sqrt{-1},\ r=1,2,\cdots,k_2-1. \tag{5b}$$

□

Lemma 2.2. We have determined unknown probabilities as follows:

$$\phi(0,j) = \phi_0\Delta_j, \quad \Delta_j := \frac{\mid D_j \mid}{\mid D \mid}, \quad j = 1, 2, \cdots, k_2 - 1, \tag{6}$$

where $\mid D_j \mid$ and $\mid D \mid$ are the determinants formed by coefficients of Eq.(4) and $\Delta_0 := 1$. □

Lemma 2.3.

(i) $\Phi(0, y)$ is explicitly given as follows, with the unknown probability ϕ_0,

$$\begin{aligned} \therefore \quad \Phi(0,y) :=& \phi_0\Phi(y) = \frac{\phi_0}{\Lambda_0(y)} \sum_{j=1}^{k_2-1} \Delta_j\{y^{k_2}G_2(G_1)^j - y^j G_2(G_1)^{k_2}\} \\ &+ \frac{\phi_0}{\Lambda_0(y)}\{V(G_1,y)y^{k_2} - G_2(G_1)^{k_2} - \Lambda_0(y)V(0,y)\} \\ &+ \frac{\phi_0}{\Lambda_0(y)} \sum_{j=1}^{k_2-1} v(0,j)\left\{y^{k_2}G_2(G_1)^j - y^j G_2(G_1)^{k_2}\right\}. \end{aligned} \tag{7}$$

$$\therefore \quad \Phi_2(x,y) := \phi_0 F_2(x,y),$$

$$\begin{aligned} F_2(x,y) :=& \{\Phi(y) - 1\}\frac{G_2(x)^{k_2}}{y^{k_2}} + \sum_{j=1}^{k_2-1} \Delta_j y^j \left\{\frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}}\right\} \\ &+ V(0,y)\frac{G_2(x)^{k_2}}{y^{k_2}} + \sum_{j=1}^{k_2-1} v(0,j)y^j \left\{\frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}}\right\}. \end{aligned} \tag{8a}$$

$$\begin{aligned} \therefore \quad F_2(x,y) =& \{\Phi(y) - 1 + V(0,y)\}\frac{G_2(x)^{k_2}}{y^{k_2}} \\ &+ \sum_{j=1}^{k_2-1} \{\Delta_j + v(0,j)\}y^j \left\{\frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}}\right\}, \end{aligned} \tag{8b}$$

$$\begin{aligned} \therefore \quad \Phi(y) =& 1 - V(0,y) + \frac{y^{k_2}}{\Lambda_0(y)} \\ &\times \left[V(G_1,y) - 1 + \sum_{j=1}^{k_2-1} \{\Delta_j + v(0,j)\}y^j \left\{\frac{G_2(x)^j}{y^j} - \frac{G_2(x)^{k_2}}{y^{k_2}}\right\}\right], \end{aligned} \tag{8c}$$

where $\Phi(y) := \Phi(0,y)/\phi_0$, and $F_2(x,y) := \Phi_2(x,y)/\phi_0$. We know that $\Phi(0,y)$ can be expressed by the form $\phi_0\Phi(y)$.

(ii) $\Phi_2(x, y)$ is given by Eq. (8b) and $\phi_0 F_2(x, y)$, where ϕ_0 should be determined from that $\Phi_1(0, 1) + \Phi_2(1, 1) = 1$. (However, $C_n \times \phi_0$ shall be determined by using the normalizing constant to be $\Pi_n(1) = 1$ later on). □

2.2 Queue-Length GF at Message Departure Epochs

In this subsection, we analyze the queue-length ditribution at departure epochs of messages from each service counter. Let,

$\pi_n(i)$: the steady-state probability that i messages are waiting in Q_n just after a class-n message has completed service at the service counter $S_n, n = 1, 2, i = 0, 1, 2, \cdots$, and

$\Pi_n(\cdot) = \Pi_n(x) := \sum_{i=0}^{\infty} \pi_n(i)x^i, |x| \leq 1, n = 1, 2.$

We first define a conditional probability and its generating function:

$q_n(i;m)$: the probability that i messages are waiting in Q_n just after a class-n message has completed service during an m (≥ 1)-busy period at $S_n, n = 1, 2$, and

$Q_n(x;m) := \sum_{i=0}^{\infty} q_n(i;m)x^i$.

Lemma 2.4. We obtain an elementary relationship between $Q_n(x;m)$ and $Q_n(x)$ such that

$$Q_n(x;m) = \frac{Q_n(x)(x^m - 1)}{x - Q_n(x)}, \quad n = 1, 2 \tag{9a}$$

which follows from the following balance equation for $\{q_n(i;m)\}$:

$$q_n(i;m) = q_n(i - m + 1) + \sum_{j=1}^{i+1} q_n(j;m)q_n(i - j + 1). \tag{9b}$$

The first term on the right-hand side of Eq. (9b) corresponds to $\{q_n(j = m;m) \equiv 1\} \times q_n(i-m+1)$ in the second term. Note that the first term $q_n(m;m) \equiv 1$ can be not included in the second term, because of $q_n(m;m) < 1$ in general. That is, we need the first term. Further, from this balance equation we obtain

$$Q_n(x;m) = x^{m-1}Q_n(x) + \{Q_n(x;m) - Q_n(0;m)\}Q_n(x)/x. \tag{9c}$$

The first term on the right-hand side follows from $\sum_{i=0}^{\infty} q_n\{i - (m-1)\}x^i = x^{m-1}\sum_{i=0}^{\infty} q_n(i)x^i$. Note that a steady busy period process exists under $\rho < 1$, $q_n(0;m) = 1$ and $q_n(i \geq 1;m) = 0$ at a stopping epoch of an m-busy period process, that is, $Q_n(0;m) = 1$. □

2.3 The Case of $(k_1 = \infty, k_2 \geq 1)$

(i) **Derivation of** $\Pi_1(x)$

First of all, we need the GF $P_1(x)$ of the queue-length distribution $\{p_1(i),\ i \geq 1\}$ at service-beginning epochs at S_1 for the derivation of $\Pi_1(x)$. That is, $P_1(x) := \sum_{i=1}^{\infty} p_1(i)x^i = P_1(x,1)$, $P_1(x,y) := \phi_0\{V(x,y) - V(0,y)\} + \sum_{n=1}^{2}\{\Phi_n(x,y) - \Phi_n(0,y)\}$. Note that we need three minus terms, $V(0,y)$ and $\Phi_n(0,y)$, $n = 1, 2$, because of $\{p_1(i), i \geq 1\}$.

$$\Pi_1(x) = C_1 \sum_{i=1}^{\infty} p_1(i)Q_1(x;i), \text{where } C_1 \text{ is the normalizing constant to be } \Pi_1(1) = 1.$$

$$\begin{aligned}
\Pi_1(x) =& C_1 Q_1(x;i)\left[\phi_0 \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\{v(i,j) - v(0,j)\} + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\{\phi_2(i,j) - \phi_2(0,j)\}\right] \\
=& \frac{C_1 Q_1(x)}{x - Q_1(x)}[\phi_0\{V(x,1) - V(0,1)\} - \phi_0\{V(1,1) - V(0,1)\} \\
&+ \Phi_2(x,1) - \Phi_2(0,1) - \{\Phi_2(1,1) - \Phi_2(0,1)\}] \\
=& \frac{C_1 Q_1(x)}{x - Q_1(x)}[\phi_0\{V(x,1) - 1\} + \Phi_2(x,1) - \Phi_2(1,1)],
\end{aligned}$$

$$\therefore \quad \Pi_1(x) = \frac{C_1 \phi_0 Q_1(x)}{x - Q_1(x)}\{V(x,1) - 1 + F_2(x,1) - F_2(1,1)\}, \tag{10}$$

where $\phi_0 F_2(x,y) = \Phi_2(x,y)$. Note that $\Phi_1(0,y) + \Phi_2(0,y) = \Phi_2(0,y)$ because of $\{p_1(i), i \geq 1\}$. The normalizing constant $C_1 \times \phi_0$ can be determined from $\Pi_1(1) = 1$ using L'Hospital's rule.

(ii) **Derivation of** $\Pi_2(y)$

We need the queue-length distribution at service-beginning epochs at S_2,$\{p_2(j), j \geq 1\}$ for the derivation of $\Pi_2(y)$.

$$P_2(y) = \sum_{j=1}^{\infty} p_2(j)y^j := P_2(0,y), \tag{11a}$$

where $P_2(0,y) := \phi_0\{V(0,y) - V(0,0)\} + \{\Phi(0,y) - \Phi(0,0)\}$.

$$\begin{aligned}
\Pi_2(y) &= \sum_{j=1}^{k_2-1} p_2(j)Q_2(y;j) + \left\{ P_2(y)/y^{k_2} - \sum_{j=1}^{k_2-1} p_2(j)y^j/y^{k_2} \right\} Q_2(y;k_2) \\
&=\phi_0 \sum_{j=1}^{k_2-1} v(0,j)Q_2(y;j) + \phi_0 \left\{ V(0,y)/y^{k_2} - \sum_{j=1}^{k_2-1} v(0,j)y^j/y^{k_2} \right\} Q_2(y;k_2) \\
&\quad + \sum_{j=1}^{k_2-1} \phi(0,j)Q_2(y;j) + \left\{ (\Phi(0,y) - \phi_0)/y^{k_2} - \sum_{j=1}^{k_2-1} \phi(0,j)y^j/y^{k_2} \right\} Q_2(y;k_2) \\
&=\frac{C_2Q_2(y)}{y - Q_2(y)}(1 - y^{-k_2})\{\Phi(0,y) - \phi_0 + \phi_0 V(0,y)\} \\
&\quad + \frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)} \left[\sum_{j=1}^{k_2-1} \{\Delta_j + v(0,j)\} \times \{y^j - 1 - y^j/y^{k_2}(y^{k_2} - 1)\} \right].
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \Pi_2(y) &=\frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)}(1 - y^{-k_2}) \times \frac{y^{k_2}}{y^{k_2} - G_2(G_1)^{k_2}} \\
&\quad \times \left[V(G_1,y) - 1 + \sum_{j=1}^{k_2-1} (\Delta_j + v(0,j))\{y^{k_2}G_2(G_1)^j - y^j G_2(G_1)^{k_2}\} \right] \\
&\quad + \frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)} \left[\sum_{j=1}^{k_2-1} \{\Delta_j + v(0,j)\}(y^{j-k_2} - 1) \right]. \qquad (11b)
\end{aligned}$$

2.3.1 Waiting Time Analysis

We denote by $W_n^*(s)$ the LST of distribution function for the waiting time of class-n messages $W_n(t), n = 1,2$. By the usual argument, the number of class-n messages at a departure epoch from S_n is equal to the number of class-n messages that arrive at Q_n during its sojourn time because of the FIFO service discipline. Therefore, we obtain

$$\therefore \quad W_n^*(s) = \Pi_n(1 - s/\lambda_n)/H_n^*(s), \; n = 1,2. \qquad (12)$$

(i) **Derivation of $W_1^*(s)$**

From (10), we have

$$\Pi_1(x) = \frac{C_1\phi_0 Q_1(x)}{x - Q_1(x)}\{V(x,1) - 1 + F_2(x,1) - F_2(1,1)\},$$

where $F_2(x,y) = \Phi_2(x,y)/\phi_0, \; F_1(x) := V(x,1) - 1 + F_2(x,1) - F_2(1,1)$. Setting $x = 1 - s/\lambda_1$ leads to $W_1^*(s)$, (it is equivalent to (12)):

$$\therefore \quad W_1^*(s) = \frac{(1-\rho_1)s}{s - \lambda_1\{1 - H_1^*(s)\}} \frac{F_1(1 - s/\lambda_1)}{-s/\lambda_1}. \qquad (13a)$$

(ii) **Derivation of $W_2^*(s)$**

From (11b), we have

$$\begin{aligned}
\Pi_2(y) =&\frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)}(1 - y^{-k_2})\{\Phi(y) - 1 + V(0,y)\} \\
&+ \frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)} \left[\sum_{j=1}^{k_2-1} \{\Delta_j + v(0,j)\}(y^{j-k_2} - 1) \right],
\end{aligned}$$

where $F_2(y) := (1-y^{-k_2})\{\Phi(y)-1+V(0,y)\}+\sum_{j=1}^{k_2-1}\{\Delta_j+v(0,j)\}(y^{j-k_2}-1)$, and $\Phi(0,y) = \phi_0\Phi(y)$. Similarly, setting $y = 1 - s/\lambda_2$ leads to $W_2^*(s)$, (it is equivalent to (12)):

$$\therefore \quad W_2^*(s) = \frac{(1-\rho_2)s}{s-\lambda_2\{1-H_2^*(s)\}}\frac{F_2(1-s/\lambda_2)}{-s/\lambda_2}. \tag{13b}$$

For the above our process to $W_n^*(s)$, i.e. "$\Phi_n(x,y) \to P_n(x,y) \to \Pi_n(\cdot) \to W_n^*(s), n = 1,2,$" we call <Top-Down Procedure> (TDP).

The above analyses of (13a) and (13b) are summarized as follows:

Theorem 2.1. For the case $(k_1 = \infty, k_2 \geq 1)$, the LSTs $W_n^*(s), n = 1,2$, are given by the top-down procedure, where

$$P_1(x,y) :=\phi_0\{V(x,y) - V(0,y)\} + \sum_{n=1}^{2}\{\Phi_n(x,y) - \Phi_n(0,y)\}, \tag{13c}$$
$$P_2(0,y) :=\phi_0\{V(0,y) - V(0,0)\} + \{\Phi(0,y) - \Phi(0,0)\}.$$

□

(iii) **Mean Waiting Time**

Denoting by $E(W_n)_K,\ n = 1,2$ the mean waiting time for class-n messages in the case of $(k_1 = \infty, k_2 = K \geq 1)$, it is given by:

Corollary 2.1.

$$E(W_n)_K = \frac{\lambda_n h_n^{(2)}}{2(1-\rho_n)} + \frac{1}{\lambda_n}\frac{F_n^{''}(1)}{2F_n^{'}(1)}, \quad F_n^m(x) \equiv \frac{d^m}{dx^m}F_n(x),\ \ n = 1,2, \tag{14}$$

where we have used $W_n^*(0) = 1$ and applied L'Hospital's rule to (11b) because of $F_n(1) = 0$. □.

For the semi-exhaustive service priority model of $(k_1 = \infty, k_2 \equiv K = 1)$, the mean waiting time formulas are given by:

Corollary 2.2.

$$E(W_1)_{K=1} =\frac{\lambda_1 h_1^{(2)}}{2(1-\rho_1)} + \frac{\lambda_2 h_2^{(2)}}{2(1-\rho_1)(1-\rho_2)^2} + \frac{(1-\rho)}{1-\rho_1}\frac{v^{(2)}}{2v}, \tag{15a}$$
$$E(W_2)_{K=1} =\frac{\lambda_1 h_1^{(2)}}{2(1-\rho_1)(1-\rho)} + \frac{\lambda_2 h_2^{(2)}}{2(1-\rho_2)}\left[1+\frac{\rho_1^2}{(1-\rho_1)(1-\rho_2)(1-\rho)}\right] + \frac{1}{1-\rho_1}\frac{v^{(2)}}{2v}. \tag{15b}$$

□

2.4 The Case of $(k_1 = k_2 = \infty)$

For this case, we have

$$\Phi_1(0,y) = \phi_0\{V(G_1(y),y) - V(0,y)\} + \Phi_2(G_1(y),0) - \Phi_2(0,0), \tag{16}$$
$$\Phi_2(x,0) = \phi_0\{V(0,G_2(x)) - V(0,0)\} + \Phi_1(0,G_2(x)) - \Phi_1(0,0), \tag{17}$$
$$\Phi_2(G_1(y),0) = \phi_0\{V(0,G_2(G_1(y)))\} + \Phi_1(0,G_2(G_1(y)) - \Phi_1(0,0). \tag{16a}$$

Here, eliminating $\Phi_2(G_1(y),0)$ from (16) and (16a) after setting $x = G_1(y)$, we obtain

$$\Phi_1(0,y) =\phi_0\{V(G_1(y),y) - V(0,y)\} + \phi_0\{V(0,G_2(G_1(y)))\} + \Phi_1\{0,G_2(G_1(y))\} - \Phi_1(0,0) - \Phi_2(0,0). \tag{16b}$$

Further, using $\Phi(0,y) := \Phi_1(0,y) + \Phi_2(0,0)$, ($\because$ the exhaustive service at S_2), we obtain

$$\begin{aligned}
&\Phi(0,y) - \Phi\{0, G_2(G_1(y))\} \\
&\quad = \phi_0\{V(G_1(y),y) - V(0,y) + V(0, G_2(G_1(y))) - 1\}. \qquad (16c) \\
&\qquad \varphi(y) := \Phi(0,y), \\
\therefore \quad &\varphi(y) - \varphi\{f(y)\} = \phi_0 g(y), \qquad (16d) \\
&\quad f(y) := G_2(G_1(y)), \\
&\quad g(y) := V(G_1(y),y) - V(0,y) + V(0, G_2(G_1(y))) - 1.
\end{aligned}$$

Using the iterative scheme (Kuczma et al. [15]), $\varphi(y)$ can be determined as follows. First, let us introduce a sequence of $\{y_i\}$,

$$\begin{aligned}
&y_{i+1} := f(y_i), \quad i = 0,1,2,\cdots \quad y_0 := y, \quad 0 \le y \le 1, \qquad (17a) \\
&g(y_i) := g\{y_i | y_0 = y\}.
\end{aligned}$$

Then, it follows from (16d) that

$$\varphi(y_i) - \varphi(y_{i+1}) = \phi_0 g(y_i), \quad i = 0,1,2,\cdots.$$

Using this relation repeatedly, we have

$$\varphi(y) = \eta + \phi_0 \sum_{i=0}^{\infty} g\{y_i | y_0 = y\}. \qquad (17b)$$

By the boundary condition, $\varphi(0) = \phi_0$, the constant η can be determined as

$$\eta = \phi_0 \left[1 - \sum_{i=0}^{\infty} g\{y_i | y_0 = 0\}\right].$$

Thus, we obtain

$$\begin{aligned}
\therefore \quad \Phi(0,y) := \varphi(y) &= \phi_0\{1 - G(0) + G(y)\}, \qquad (18) \\
G(y) &:= \sum_{i=0}^{\infty} g\{y_i | y_0 = y\}. \qquad (18a)
\end{aligned}$$

Similarly, we know that $\Phi(x,0) := \Phi_2(x,0) + \Phi_1(0,0)$ and $\Phi(0,y)$ lead to the top-down procedure, which are summarized in Theorem 2.1. (This suggests us an important problem, see Remark 4.3).

Corollary 2.3. For the exhaustive service priority model of $(k_1 = \infty, k_2 \equiv K = \infty)$, the mean waiting time formulas are given by:

$$\begin{aligned}
E(W_1)_{K=\infty} = &\frac{\lambda_1 h_1^{(2)}}{2(1-\rho_1)} + \frac{\rho_2^2 \lambda_1 h_1^{(2)} + (1-\rho_1)^2 \lambda_2 h_2^{(2)}}{2(1-\rho_1)(1-\rho_2)(1-\rho+2\rho_1\rho_2)} \qquad (19a) \\
&+ \frac{\rho_2 \rho + (1-\rho_2)(1-\rho)}{1-\rho+2\rho_1\rho_2} \frac{v^{(2)}}{2v}, \\
E(W_2)_{K=\infty} = &\frac{\lambda_2 h_2^{(2)}}{2(1-\rho_2)} + \frac{(1-\rho_2)^2 \lambda_1 h_1^{(2)} + \rho_1^2 \lambda_2 h_2^{(2)}}{2(1-\rho_1)(1-\rho_2)(1-\rho+2\rho_1\rho_2)} \qquad (19b) \\
&+ \frac{1-\rho_2}{1-\rho+2\rho_1\rho_2} \frac{v^{(2)}}{2v}.
\end{aligned}$$

□

2.5 The Case of $(k_1 = \infty, k_2 = 1)$

For $k_2 = 1$, we have the following equation:

$$\therefore \quad \Pi_1(x) = \frac{C_1\phi_0 Q_1(x)}{x - Q_1(x)}\{V(x,1) - 1 + F_2(x,1) - F_2(1,1)\}, \tag{20a}$$

where $F_2(x,y) = \Phi_2(x,y)/\phi_0,\ F_1(x) := V(x,1) - 1 + F_2(x,1) - F_2(1,1)$.

$$\begin{aligned}\therefore \quad \Pi_2(y) =& \frac{C_2\phi_0 Q_2(y)}{y - Q_2(y)}(1 - y^{-1}) \times \frac{y}{y - G_2(G_1)} \times [V(G_1, y) - 1] \\ =& \frac{1-\rho}{\lambda_2 v}\frac{Q_2(y)}{y - Q_2(y)}(y-1)\frac{V\{G_1(y), y\} - 1}{y - G_2(G_1(y))}.\end{aligned} \tag{20b}$$

Setting $x = 1 - s/\lambda_1$ and $y = 1 - s/\lambda_2$, $\Pi_1(x)$ and $\Pi_2(y)$, respectively, lead to the LST $W_n^*(s), n = 1,2$ for the case of $k_2 = 1$:

Theorem 2.2. For the semi-exhaustive service priority model, the LSTs $W_n^*(s), n = 1,2$ are given by:

$$W_1^*(s)_{K=1} = \frac{(1-\rho_1)s}{s - \lambda_1\{1 - H_1^*(s)\}}\frac{F_1(1 - s/\lambda_1)}{-s/\lambda_1}, \tag{21a}$$

$$W_2^*(s)_{K=1} = \frac{1-\rho}{\lambda_2 v} \cdot \frac{1 - \theta_v^*(s)}{sE(\theta_v)} \cdot \frac{s\{1 - \lambda_2 E(D_2)\}}{s - \lambda_2 + \lambda_2 D_2^*(s)} \cdot \frac{s(1-\rho_2)}{s - \lambda_2 + \lambda_2 H_2(s)}, \tag{21b}$$

where

$$\begin{aligned}\theta_v^*(s) :=& V^*(\sigma_1), \quad \sigma_1 := s + \lambda_1 - \lambda_1\Theta_1^*(s), \\ D_2^*(s) :=& \Theta_2^*\{\lambda_1 - \lambda_1\Theta_1^*(s)\}.\end{aligned} \tag{21c}$$

For the right-hand side of Eq. (21b), some probabilistic meanings for $W_2^*(s)_{K=1}$ can be given by the decomposition (law) property in Theorem 4.2. □

3 Three-class Priority Queues $(N = 3)$

This section presents three-class priority queues with general decreasing service of a parameter $(k_1 = \infty, k_2 \geq 1, k_3 \geq 1)$. Each queue Q_n with infinite buffer capacity has a service counter for class-n messages $S_n,\ n = 1,2,3$. Messages in Q_1, Q_2 and Q_3 are served by a single server, who serves all messages in accordance to the following priority discipline; after switching over to S_1, the server serves all messages in Q_1 until it becomes empty, i.e. the exhaustive service $(k_1 = \infty)$. Just after service completion at S_1, the server switches over to S_2 and continues serving messages in Q_2 until either it becomes empty, or the number of messages decreases to k_2 less than that found upon its arrival at S_2, whichever occurs first, i.e. the k_2-decreasing service. Class-3 messages are served next, if any is present, according to the k_3-decreasing service.

We introduce the following notation:

$$r_n := \lambda_n/\lambda,\ n = 1,2,3, \qquad \lambda := \lambda_1 + \lambda_2 + \lambda_3,$$

$$\rho_n := \lambda_n h_n,\ n = 1,2,3, \qquad \rho := \rho_1 + \rho_2 + \rho_3, \qquad \rho_n^+ = \sum_{i=1}^{n}\rho_i,$$

$$h := r_1 h_1 + r_2 h_2 + r_3 h_3, \qquad h^{(2)} := r_1 h_1^{(2)} + r_2 h_2^{(2)} + r_3 h_3^{(2)},$$

and

$$
\begin{aligned}
&Q_n(x) := H_n^*\{\lambda_n(1-x)\}, \quad n = 1,2,3,\\
&Q_v(x,y,z) := V^*\{\lambda_1(1-x)+\lambda_2(1-y)+\lambda_3(1-z)\} = V(x,y,z),\\
&G_1(y,z) := \Theta_1^*\{\lambda_2(1-y)+\lambda_3(1-z)\}, \quad \Theta_1^*(s) = H_1^*\{s+\lambda_1(1-\Theta_1^*(s))\},\\
&G_2(x,z) := \Theta_2^*\{\lambda_1(1-x)+\lambda_3(1-z)\}, \quad \Theta_2^*(s) = H_2^*\{s+\lambda_2(1-\Theta_2^*(s))\},\\
&G_3(x,y) := \Theta_3^*\{\lambda_1(1-x)+\lambda_2(1-y)\}, \quad \Theta_3^*(s) = H_3^*\{s+\lambda_3(1-\Theta_3^*(s))\}.
\end{aligned}
$$

The queue-length probability defined at message departure epochs, $\pi_n(i)$, introduced in 2.2, is also used in this section for $n = 1,2,3$. Its generating function is defined as follows.

$$\Pi_n(\cdot) = \Pi_n(x) := \sum_{i=0}^{\infty} \pi_n(i)x^i, \;\; |x| \le 1, n = 1,2,3.$$

Also, let $\pi_n(i,j,k)$ denote the probability that i, j, and k messages are waiting in queues Q_1, Q_2, and Q_3, respectively, at message departure epochs at S_n. Then,

$$\Pi_n(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \pi_n(i,j,k)x^i y^j z^k, \quad |x|,|y|,|z| \le 1, n = 1,2,3.$$

Similarly, assume that $p_n(i,j,k)$ represents the probability that i, j, and k messages are waiting in Q_1, Q_2, and Q_3, respectively, at service beginning epochs at S_n. Then,

$$P_n(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_n(i,j,k)x^i y^j z^k, \quad |x|,|y|,|z| \le 1, n = 1,2,3.$$

The queue length probability at service completion epochs, $\phi_n(i,j)$, defined in Section 2 is also used, but extended as $\phi_n(i,j,k)$ for $N = 3$.

3.1 Queue-Length GF at Service-Completion Epochs ($N = 3$)

First, let us define

$$\Phi_n(x,y,z) := \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_n(i,j,k)x^i y^j z^k, \quad |x|,|y|,|z| \le 1, \quad n = 1,2,3. \tag{22}$$

$\Phi(x,y,z) := \sum_{n=1}^{3} \Phi_n(x,y,z)$, and $\phi_0 := \Phi(0,0,0)$.

For $\Phi_n(x,y,z), n = 1,2,3$, we obtain the following functional relationship:

$$
\begin{aligned}
\Phi_1(0,y,z) =& \phi_0\{V(G_1(y,z),y,z) - V(0,y,z)\}\\
&+ \Phi_2(G_1(y,z),y,z) - \Phi_2(0,y,z) + \Phi_3(G_1(y,z),y,z) - \Phi_3(0,y,z).
\end{aligned} \tag{22a}
$$

The term $G_1(y,z)$ is used for the exhaustive service at S_1. The movement of a moving single server (which corresponds to the control-point used in processor and communication systems) may be $V \to S_1, S_2 \to S_1$ and $S_3 \to S_1$. For the term $\Phi_2(x,y,z)$, the movement of a single server may be as follows, i.e. $S_1 \to S_2, S_3 \to S_2$ and $V \to S_2$.

$$
\begin{aligned}
\Phi_2(x,y,z) =& \{\Phi(0,y,z) - \Phi(0,0,z)\}\frac{G_2(x,z)^{k_2}}{y^{k_2}}\\
&+ \sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty} \phi(0,j,k)y^j z^k \left[\frac{G_2(x,z)^j}{y^j} - \frac{G_2(x,z)^{k_2}}{y^{k_2}}\right]
\end{aligned}
$$

$$+ \phi_0\{V(0,y,z) - V(0,0,z)\}\frac{G_2(x,z)^{k_2}}{y^{k_2}} \tag{22b}$$
$$+ \phi_0 \sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty} v(0,j,k)y^j z^k \left[\frac{G_2(x,z)^j}{y^j} - \frac{G_2(x,z)^{k_2}}{y^{k_2}}\right],$$

where $\Phi(0,y,z) := \sum_{n=1}^{3}\Phi_n(0,y,z)$. Note that it can't be allowed to move such as $V \to S_2$, i.e. $V(0,0,z) \overset{\times}{\longrightarrow} \Phi_2(x,y,z)$.

For $\Phi_3(x,y,z)$, the movement of a single server may be as follows, i.e. $V \to S_3, S_1 \to S_3$ and $S_2 \to S_3$.

$$\Phi_3(x,y,z) = \phi_0\{V(0,0,z) - V(0,0,0)\}\frac{G_3(x,y)^{k_3}}{z^{k_3}} - \phi_0\sum_{k=1}^{k_3-1} v(0,0,k)z^k\frac{G_3(x,y)^{k_3}}{z^{k_3}}$$
$$+ \phi_0\sum_{k=1}^{k_3-1} v(0,0,k)z^k\frac{G_3(x,y)^{k}}{z^{k}} + \{\Phi(0,0,z) - \phi_0\}\frac{G_3(x,y)^{k_3}}{y^{k_3}} \tag{22c}$$
$$- \sum_{k=1}^{k_3-1}\phi(0,0,k)z^k\frac{G_3(x,y)^{k_3}}{z^{k_3}} + \sum_{k=1}^{k_3-1}\phi(0,0,k)z^k\frac{G_3(x,y)^{k}}{z^{k}},$$

where $\phi_0 := \phi(0,0,0)$, and $V(0,0,0) \equiv 0$, ($\because$ the MV rule).

After setting $x = G_1(y,z)$ for $\Phi_2(x,y,z)$ in Eq.(22b) and $\Phi_3(x,y,z)$ in Eq.(22c), we sum three equations (22a), (22b) and (22c) for eliminating $\Phi_2(G_1,y,z)$ and $\Phi_3(G_1,y,z)$ on the right-hand side of Eq. (22a) ($G_1 \equiv G_1(y,z)$). The above equations are rearranged as follows:

$$\Phi(0,y,z)\left[1 - \frac{G_2\{G_1(y,z),z\}^{k_2}}{y^{k_2}}\right] + \sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}\phi(0,j,k)y^j z^k\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_2(G_1,z)^{j}}{y^{j}}\right]$$
$$+ \Phi(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \sum_{k=1}^{k_3-1}\phi(0,0,k)z^k\left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1,y)^{k}}{z^{k}}\right]$$
$$= -\phi_0 V(0,y,z)\left[1 - \frac{G_2\{G_1,z\}^{k_2}}{y^{k_2}}\right] + \phi_0\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty} v(0,j,k)y^j z^k\left[\frac{G_2(G_1,z)^{j}}{y^{j}} - \frac{G_2(G_1,z)^{k_2}}{y^{k_2}}\right]$$
$$- \phi_0 V(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \tag{23}$$
$$+ \phi_0\sum_{k=1}^{k_3-1} v(0,0,k)z^k\left[\frac{G_3(G_1,y)^{k}}{z^{k}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \phi_0 V(G_1,y,z) - \phi_0\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}.$$

Finally, we obtain the basic functional equation with unknown functions, $\Phi(0,y,z)$ and $\Phi(0,0,z)$ and unknown probabilities, $\phi(0,j,k)$ and $\phi(0,0,k)$, where $j \le k_2$ and $k \le k_3$.

This is the starting-point of our queueing analysis, which consists of the following three steps:

Step 1: Using the basic functional equation, we need determine the unknown functions $\Phi(0,y,z)$, $\Phi(0,0,z)$ and unknown probabilities $\phi(0,j,k), \phi(0,0,k)$.

Step 2: Using the determined $\widehat{\Phi}(0,y,z)$ and $\widehat{\Phi}(0,0,z)$, we can express $\Phi_2(x,y,z)$, $\Phi_3(x,y,z)$ and $\Phi(0,y,z)$.

Step 3: Using $\Phi_2(x,y,z), \Phi_3(x,y,z)$ and $\Phi(0,y,z)$, we can express $P_n(x,y,z), n = 1,2,3$, by which we can obtain $P_n(\cdot), \Pi_n(\cdot)$ and $W_n^*(s), n = 1,2,3$.

According to the above three steps, we proceed to the final goal, i.e. $W_n^*(s), n = 1,2,3$. For

Step 2, we obtain the following equations:

$$\Phi_1(0,y,z) = \phi_0\{V(G_1(y,z),y,z) - V(0,y,z)\} + \Phi_2\{G_1(y,z),0,z\} - \Phi_2(0,0,z) + \Phi_3\{G_1(y,z),y,0\} - \Phi_3(0,y,0), \quad (22a)$$

where we have put $x = G_1(y,z)$ ($\because$ the exhaustive service for class-1 messages).

The other functions $\Phi_2(x,y,z)$ and $\Phi_3(x,y,z)$ are given by Eqs. (22b), (22c), respectively. We know that $\Phi_n(x,y,z), n = 2,3$ can be expressed by $\Phi(0,y,z)$ and $\Phi(0,0,z)$ on the right-hand side of Eqs. (22b) and (22c), which are determined from the basic functional equation. Note that $V(x,y,z)$ and $G_n(\cdot), n = 1,2,3$ are known functions. For the derivation of $\Pi_n(\cdot), n = 1,2,3$, we need the following $P_n(x,y,z), n = 1,2,3$:

For Step 3, we obtain the following equations:

$$P_1(x,y,z) := \phi_0\{V(x,y,z) - V(0,y,z)\} + \sum_{n=2}^{3}\{\Phi_n(x,y,z) - \Phi_n(0,y,z)\}, \quad (24a)$$

$$P_2(0,y,z) := \phi_0\{V(0,y,z) - V(0,0,z)\} + \sum_{n=1}^{3}\{\Phi_n(0,y,z) - \Phi_n(0,0,z)\}, \quad (24b)$$

$$P_3(0,0,z) := \phi_0\{V(0,0,z) - V(0,0,0)\} + \sum_{n=1}^{3}\{\Phi_n(0,0,z) - \Phi_n(0,0,0)\}. \quad (24c)$$

Therefore, we know that $\Pi_n(\cdot), n = 1,2,3$ are also expressed by $\Phi(0,y,z)$ and $\Phi(0,0,z)$ using $P_n(x,y,z),\ n = 1,2,3$ and Lemma 2.4. That is, $\Phi_n(x,y,z) \to P_n(x,y,z) \to P_n(\cdot) \to \Pi_n(\cdot) \to W_n^*(s), n = 1,2,3$.

3.2 The Case of $(k_1 = \infty, k_2 = k_3 = 1)$

For $k_2, k_3 = 1$, $\sum_{j=1}^{k_2-1}(\sim) = \sum_{k=1}^{k_3-1}(\sim) = 0$ in Eq. (23), so we have the following functional equation:

$$\begin{aligned}\Phi(0,y,z)z\,[y - G_2\{G_1(y,z),z\}] &+ \Phi(0,0,z)\,[zG_2(G_1,z) - yG_3(G_1,y)] \\ = &- \phi_0 V(0,y,z)z\,[y - G_2\{G_1,z\}] + \phi_0 V(0,0,z)\,[y - G_3\{G_1(y,z),y\}] \\ &+ \phi_0 y\,\{zV(G_1,y,z) - G_3(G_1,y)\}\,. \end{aligned} \quad (25)$$

Using the zero $y := \delta(z)$ for $y - G_2\{G_1(y,z),z\} = 0$, we get $\Phi(0,0,z)$ with the unknown probability ϕ_0:

$$\begin{aligned}&\Phi(0,0,z)\,[zG_2(G_1,z) - \delta(z)G_3(G_1,\delta(z))] \\ &= \phi_0 V(0,0,z)\,[\delta(z) - G_3\{G_1(\delta(z),z),\delta(z)\}] + \phi_0\delta(z)\,\{zV(G_1,\delta(z),z) - G_3(G_1,\delta(z))\}\,,\end{aligned}$$

where $G_1 \equiv G_1(\delta(z),z)$.

$$\begin{aligned}\therefore\quad \widehat{\Phi}(0,0,z) &:= \phi_0\Phi(0,0,z) \\ &= \phi_0 V(0,0,z)\,[\delta(z) - G_3\{G_1(\delta(z),z),\delta(z)\}]\,/\,[zG_2(G_1,z) - \delta(z)G_3(G_1,\delta(z))] \\ &\quad + \phi_0\delta(z)\,\{zV(G_1,\delta(z),z) - G_3(G_1,\delta(z))\}\,/\,[zG_2(G_1,z) - \delta(z)G_3(G_1,\delta(z))]\,.\end{aligned} \quad (26)$$

Here, we denote by $\widehat{\Phi}(0,0,z)$ which has been determined, while $\Phi(0,0,z)$ is unknown and still undetermined. Furthermore, using the $\widehat{\Phi}(0,0,z)$, we also obtain $\widehat{\Phi}(0,y,z)$ with unknown probability ϕ_0.

$$\begin{aligned}\therefore\quad \widehat{\Phi}(0,y,z)\cdot z\,[y - G_2\{G_1(y,z),z\}] &+ \widehat{\Phi}(0,0,z)\,[zG_2(G_1,z) - yG_3(G_1,y)] \\ = &- \phi_0 V(0,y,z)z\,[y - G_2\{G_1,z\}] + \phi_0 V(0,0,z)\,[y - G_3\{G_1(y,z),y\}] \\ &+ \phi_0 y\,\{zV(G_1,y,z) - G_3(G_1,y)\} - \phi_0 F(0,0,z)\,[zG_2(G_1,z) - yG_3(G_1,y)]\,.\end{aligned} \quad (27)$$

Meanwhile, $\Phi_n(x,y,z), n = 2,3$ can be expressed by using $\widehat{\Phi}(0,y,z)$, $\widehat{\Phi}(0,0,z)$ and Eqs. (22b), (22c).

Remark 3.1. For Eq. (22a), the other approach to $\widehat{\Phi}_1(0,y,z)$ is given as follows:

$$\begin{aligned}
&\{\Phi_2(x,y,z)+\Phi_3(x,y,z)\}_{x=0}\\
&=\widehat{\Phi}(0,y,z)\frac{G_2(0,z)}{y}-\widehat{\Phi}(0,0,z)\left\{\frac{G_2(0,z)}{y}-\frac{G_3(0,y)}{z}\right\}\\
&\quad+\phi_0V(0,y,z)\frac{G_2(0,z)}{y}+\phi_0V(0,0,z)\left\{\frac{G_3(0,y)}{z}-\frac{G_2(0,z)}{y}\right\}-\phi_0\frac{G_3(0,y)}{z}.\\
\therefore\quad &\widehat{\Phi}_1(0,y,z)=\widehat{\Phi}(0,y,z)-\{\Phi_2(x,y,z)+\Phi_3(x,y,z)\}_{x=0}.
\end{aligned}$$

3.2.1 Queue-Length GF at Message Departure Epochs

(i) **Derivation of** $\Pi_1(x)$

The GFs $P_1(x,y,z), P_2(0,y,z)$ and $P_3(0,0,z)$ used in the following (i) $\sim$ (iii) are given by (24a), (24b) and (24c), respectively.

$$\begin{aligned}
P_1(x) :=&\sum_{i=1}^{\infty}p_1(i)x^i := P_1(x,1,1)\\
&=\phi_0\{V(x,1,1)-V(0,1,1)\}+\sum_{n=2}^{3}\{\Phi_n(x,1,1)-\Phi_n(0,1,1)\},
\end{aligned}\tag{28a}$$

$$\begin{aligned}
\Pi_1(x) =&C_1\sum_{i=1}^{\infty}p_1(i)Q_1(x;i)\\
=&\frac{C_1Q_1(x)}{x-Q_1(x)}\sum_{i=1}^{\infty}p_1(i)(x^i-1)=\frac{C_1Q_1(x)}{x-Q_1(x)}\{P_1(x)-1\},\\
\Pi_1(1) =&1=C_1P'_{1,x}(1)/(1-\rho_1),\quad \therefore C_1=(1-\rho_1)/P'_{1,x}(1),\\
\therefore\quad \Pi_1(x) =&\frac{(1-\rho_1)Q_1(x)}{x-Q_1(x)}\frac{\{P_1(x)-1\}}{P'_{1,x}(1)},\quad P'_{1,x}(1):=\left[\frac{d}{dx}P_1(x)\right]_{x=1}.
\end{aligned}\tag{28b}$$

(ii) **Derivation of** $\Pi_2(y)$

$$P_2(y) :=\sum_{j=1}^{\infty}p_2(j)y^j := P_2(0,y,1),\tag{29a}$$

$$\begin{aligned}
\Pi_2(y) =&C_2\sum_{j=1}^{k_2-1}p_2(j)Q_2(y;j)+C_2\left\{P_2(y)/y^{k_2}-\sum_{j=1}^{k_2-1}p_2(j)y^j/y^{k_2}\right\}Q_2(y;k_2)\\
=&C_2\sum_{j=1}^{k_2-1}p_2(j)\{Q_2(y;j)-y^j/y^{k_2}Q_2(y;k_2)\}+C_2P_2(y)/y^{k_2}Q_2(y;k_2)\\
=&\frac{C_2Q_2(y)}{y-Q_2(y)}\sum_{j=1}^{k_2-1}p_2(j)\{y^{j-k_2}-1\}+C_2P_2(y)\frac{Q_2(y)}{y-Q_2(y)}(1-y^{-k_2}).
\end{aligned}$$

$$\therefore\quad \Pi_2(y)=C_2P_2(y)\frac{Q_2(y)}{y-Q_2(y)}(y-1)/y,\qquad (\because\ k_2=1).\tag{29b}$$

(iii) **Derivation of** $\Pi_3(z)$

$$P_3(z) := \sum_{k=1}^{\infty} p_3(k) z^k := P_3(0,0,z), \tag{30a}$$

$$\Pi_3(z) = C_3 \sum_{k=1}^{k_3-1} p_3(k) Q_3(z;k) + C_3 \left\{ P_3(z)/z^{k_3} - \sum_{k=1}^{k_3-1} p_3(k) z^k / z^{k_3} \right\} Q_3(z;k_3),$$

$$\therefore \quad \Pi_3(z) = C_3\{\phi_0 V(0,0,z) + \Phi(0,0,z) - \phi_0\} \frac{Q_3(z)(z-1)/z}{z - Q_3(z)}, \quad (\because\ k_3 = 1). \tag{30b}$$

3.2.2 Waiting Time Analysis

From the above analyses for the semi-exhaustive service priority model ($k_1 = \infty, k_2 = k_3 = 1 = K$), the final result for $W_n^*(s), n = 1,2,3$ are given by:

Theorem 3.1.

$$W_1^*(s) = \frac{(1-\rho_1)s}{s - \lambda_1\{1 - H_1^*(s)\}} \frac{P_1(1 - s/\lambda_1, 1, 1)}{P'_{1,x}(1,1,1)s}, \tag{31a}$$

$$W_2^*(s) = \frac{(1-\rho_2)s}{s - \lambda_2\{1 - H_2^*(s)\}} \frac{P_2(0, 1 - s/\lambda_2, 1)}{P'_{2,y}(0,1,1)s}, \tag{31b}$$

$$W_3^*(s) = \frac{(1-\rho_3)s}{s - \lambda_3\{1 - H_3^*(s)\}} \frac{P_3(0, 0, 1 - s/\lambda_3)}{P'_{3,z}(0,0,1)s}, \tag{31c}$$

where $P_1(x,y,z), P_2(0,y,z)$, and $P_3(0,0,z)$ are given by (24a), (24b) and (24c), respectively. □

(iv) **Mean Waiting Time**

For the semi-exhaustive service priority model ($k_1 = \infty, k_2 = k_3 = 1 = K$), the mean waiting time formulas are given by:

Corollary 3.1.

$$E(W_1)_{K=1} = \frac{\lambda_1 h_1^{(2)}}{2(1-\rho_1)} + \frac{1}{2(1-\rho_1)} \sum_{i=2}^{3} \frac{\lambda_i h_i^{(2)}}{(1-\rho_i)^2} + \frac{1-\rho}{1-\rho_1} \frac{v^{(2)}}{2v}, \tag{32a}$$

$$\begin{aligned} E(W_2)_{K=1} =& \frac{\lambda_2 h_2^{(2)}}{2(1-\rho_2)} \left[1 + \frac{\rho_1^2}{(1-\rho_1)(1-\rho_2^+)(1-\rho_2)} \right] \\ &+ \frac{1}{2(1-\rho_1)(1-\rho_2^+)} \left[\lambda_1 h_1^{(2)} + \frac{\lambda_3 h_3^{(2)}}{(1-\rho_3)^2} \right] + \frac{1-\rho}{(1-\rho_1)(1-\rho_2^+)} \frac{v^{(2)}}{2v}, \end{aligned} \tag{32b}$$

$$\begin{aligned} E(W_3)_{K=1} =& \frac{\lambda_3 h_3^{(2)}}{2(1-\rho_3)} \left[1 + \frac{\rho_2^+}{(1-\rho_2^+)(1-\rho)(1-\rho_3)} \right] \\ &+ \frac{1-\rho}{2(1-\rho_2^+)(1-\rho)} \{\lambda_1 h_1^{(2)} + \lambda_3 h_3^{(2)}\} + \frac{1-\rho}{(1-\rho_2^+)(1-\rho)} \frac{v^{(2)}}{2v}. \end{aligned} \tag{32c}$$

□

3.3 The Case of $(k_1 = \infty, k_2 = k_3 = \infty)$

We consider GFs $\Pi_n(x,y,z), n = 1,2,3$ for the exhaustive service priority model. Then, the following functional relations are derived, where $Q_n \equiv Q_n(x,y,z),\ n = 1,2,3$:

$$\begin{aligned} \Pi_1(x,y,z) =& \{\Pi_1(x,y,z) - \Pi_1(0,y,z)\}Q_1/x + \{\Pi_2(x,0,z) - \Pi_2(0,0,z)\}Q_1/x \\ &+ \{\Pi_3(x,y,0) - \Pi_3(0,y,0)\}Q_1/x + \pi_0\{Q_v(x,y,z) - Q_v(0,y,z)\}Q_1/x, \end{aligned} \tag{33a}$$

where $\pi_0 := \sum_{n=1}^{3} \Pi_n(0,0,0)$. We use the minus terms, $\Pi_1(0,y,z), \Pi_2(0,0,z), \Pi_3(0,y,0)$, and $Q_v(0,y,z)$. (These minus terms guarantee the exhaustive service).

$$\begin{aligned}\Pi_2(x,y,z) =& \{\Pi_2(x,y,z) - \Pi_2(x,0,z)\}Q_2/y + \{\Pi_1(0,y,z) - \Pi_1(0,0,z)\}Q_2/y \\ &+ \{\Pi_3(0,y,0) - \Pi_3(0,0,0)\}Q_2/y + \pi_0\{Q_v(0,y,z) - Q_v(0,0,z)\}Q_2/y, \quad (33b)\\ \Pi_3(x,y,z) =& \{\Pi_3(x,y,z) - \Pi_3(x,y,0)\}Q_3/z + \{\Pi_1(0,0,z) - \Pi_1(0,0,0)\}Q_3/z \\ &+ \{\Pi_2(0,0,z) - \Pi_2(0,0,0)\}Q_3/z + \pi_0\{Q_v(0,0,z) - Q_v(0,0,0)\}Q_3/z. \quad (33c)\\ \Pi_1(x,y,z)\{x - Q_1\}/Q_1 + \Pi_1(0,y,z) =& \{\Pi_2(x,0,z) - \Pi_2(0,0,z)\} \\ &+ \{\Pi_3(x,y,0) - \Pi_3(0,y,0)\} + \pi_0\{Q_v(x,y,z) - Q_v(0,y,z)\}, \quad (34a)\\ \Pi_2(x,y,z)\{y - Q_2\}/Q_2 + \Pi_2(x,0,z) =& \{\Pi_1(0,y,z) - \Pi_1(0,0,z)\} \\ &+ \{\Pi_3(0,y,0) - \Pi_3(0,0,0)\} + \pi_0\{Q_v(0,y,z) - Q_v(0,0,z)\}, \quad (34b)\\ \Pi_3(x,y,z)\{z - Q_3\}/Q_3 + \Pi_3(x,y,0) =& \{\Pi_1(0,0,z) - \Pi_1(0,0,0)\} \\ &+ \{\Pi_2(0,0,z) - \Pi_2(0,0,0)\} + \pi_0\{Q_v(0,0,z) - Q_v(0,0,0)\}. \quad (34c)\end{aligned}$$

The right-hand side of Eq.(34a) is expressed by $\varphi_1(x,y,z)$. Similarly, the right-hand sides of (34b) and (34c) can be expressed by $\varphi_2(y,z)$ and $\varphi_3(z)$, respectively. Using the zero, $x = \bar{\delta}_1(y,z)$ for $[x - Q_1(x,y,z) = 0]$, we obtain $\Pi_1(0,y,z) = \varphi_1\{\bar{\delta}_1(y,z), y, z\}$, That is, $\Pi_1(x,y,z)\{x - Q_1\}/Q_1 = \varphi_1(x,y,z) - \varphi_1(\bar{\delta}_1, y, z)$. Similarly, $\Pi_2(x,y,z)\{y - Q_2\}/Q_2 = \varphi_2(x,z) - \varphi_2(\bar{\delta}_2(x,z), z)$, $\Pi_3(x,y,z)\{z - Q_3\}/Q_3 = \varphi(z) - \varphi_3(\bar{\delta}_3(x,y))$. Adding both sides of (34a)∼(34c) lead to the following invariant relation. ($Q_v(0,0,0) \equiv 0$, ∵ MV rule).

Lemma 3.1.

$$\sum_{n=1}^{3} \Pi_n(x,y,z)\{\bar{z}_n(x,y,z) - Q_n(x,y,z)\}/Q_n(x,y,z) = \pi_0\{Q_v(x,y,z) - 1\}, \quad (35)$$

where $\bar{z}_n(x,y,z) - Q_n(x,y,z)$ corresponds to $x - Q_1,\ y - Q_2$, and $z - Q_3$, respectively. □

Note that n=3 here can be extended to the general $N \geq 3$. Accordingly, our remaining analysis and the formula on $W_n^*(s)$ and $E(W_n)_{K=\infty}$ are presented in Subsection 4.1.

3.4 The Case of $(k_1 = \infty, k_2 = 1, k_3 \geq 1)$

For the case of ($k_1 = \infty, k_2 = 1, k_3 \geq 1$), the basic functional equation is given by:

$$\begin{aligned}\Phi(0,y,z)&\left[1 - \frac{G_2\{G_1(y,z),z\}}{y}\right] + \Phi(0,0,z)\left[\frac{G_2(G_1,z)}{y} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \\ &+ \sum_{k=1}^{k_3-1} \phi(0,0,k)z^k\left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1,y)^k}{z^k}\right] \qquad (36)\\ =& -\phi_0 V(0,y,z)\left[1 - \frac{G_2\{G_1,z\}}{y}\right] - \phi_0 V(0,0,z)\left[\frac{G_2(G_1,z)}{y} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \\ &+ \phi_0 \sum_{k=1}^{k_3-1} v(0,0,k)z^k\left[\frac{G_3(G_1,y)^k}{z^k} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \\ &+ \phi_0\left\{V(G_1,y,z) - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right\}.\end{aligned}$$

Setting $G_2\{G_1(y,z),z\}/y = G_3(G_1,y)^{k_3}/z^{k_3} = 1$, we obtain the zeros, $y = \delta(\omega_s)$ and $z = \omega_s, s = 1,2,\cdots,k_3 - 1$. Using the zeros, we obtain the following simultaneous linear equations.

Note that $V(x,y,z), v(i,j,k)$ and $G_n, n=1,2,3$ are known functions:

$$\sum_{k=1}^{k_3-1}\left[(\omega_s)^k - G_3\{G_1(\delta(\omega_s),\omega_s),\delta(\omega_s)\}^k\right]\phi(0,0,k)$$
$$= \phi_0 \sum_{k=1}^{k_3-1} v(0,0,k)\left[G_3\{G_1(\delta(\omega_s),\omega_s),\delta(\omega_s)\}^k - (\omega_s)^k\right] \tag{37a}$$
$$+ \phi_0\{V(G_1(\delta(\omega_s),\omega_s),\delta(\omega_s),\omega_s) - 1\}, \quad s=1,2,\cdots,k_3-1.$$

$$\sum_{k=1}^{k_3-1}\alpha\{\delta(\omega_s),\omega_s\}\phi(0,0,k) = \phi_0\{\alpha_0(\delta(\omega_s),\omega_s)-1\}, \ \ s=1,2,\cdots,k_3-1. \tag{37b}$$

$$\therefore \quad \phi(0,0,k) := \phi_0\Delta_k, \quad \Delta_k := \frac{|D_k|}{|D|}, \quad k=1,2,\cdots,k_3-1. \tag{38}$$

Using the above determined $\widehat{\Phi}(0,0,z)$, we also obtain $\widehat{\Phi}(0,y,z)$ such as

$$\widehat{\Phi}(0,y,z)\left[1-\frac{G_2\{G_1(y,z),z\}}{y}\right] + \widehat{\Phi}(0,0,z)\left[\frac{G_2(G_1,z)}{y} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]$$
$$+\phi_0\sum_{k=1}^{k_3-1}\Delta_k z^k\left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1,y)^k}{z^k}\right] \tag{39}$$
$$= -\phi_0 V(0,y,z))\left[1-\frac{G_2\{G_1,z\}}{y}\right] - \phi_0 V(0,0,z)\left[\frac{G_2(G_1,z)}{y} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]$$
$$+\phi_0\sum_{k=1}^{k_3-1} v(0,0,k)z^k\left[\frac{G_3(G_1,y)^k}{z^k} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \phi_0\left\{V(G_1,y,z) - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right\}.$$

That is, we have obtained explicit expressions for $\widehat{\Phi}(0,y,z)$ and $\widehat{\Phi}(0,0,z)$ with the unknown probability ϕ_0. □

Next, we have to go to Step 2 and Step 3 in 3.5 , at which we can derive $P_n(x,y,z), P_n(\cdot), \Pi_n(\cdot)$ and $W_n^*(s), n=1,2,3$.

Therefore, from the above analyses, the final result of $W_n^*(s), n=1,2,3$ for the case of $(k_1=\infty, k_2=1, k_3\geq 1)$ is summarized as follows:

Theorem 3.2. For the case of $(k_1=\infty, k_2=1, k_3\geq 1)$, the LSTs $W_n^*(s), n=1,2,3$ are given by the top-down procedure, where $P_1(x,y,z), P_2(0,y,z)$ and $P_3(0,0,z)$ are given by (24a), (24b) and (24c), respectively. □

3.5 The Case of $(k_1=\infty, k_2, k_3\geq 1)$

As derived in Subsection 3.1, the basic functional equation with unknown functions $\Phi(0,y,z)$ and $\Phi(0,0,z)$ is given by:

$$\Phi(0,y,z)\left[1-\frac{G_2\{G_1(y,z),z\}^{k_2}}{y^{k_2}}\right] + \sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}\phi(0,j,k)y^j z^k\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_2(G_1,z)^j}{y^j}\right]$$
$$+\Phi(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \sum_{k=1}^{k_3-1}\phi(0,0,k)z^k\left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1,y)^k}{z^k}\right]$$

$$
\begin{aligned}
=&-\phi_0 V(0,y,z)\left[1-\frac{G_2\{G_1,z\}^{k_2}}{y^{k_2}}\right]+\phi_0\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}v(0,j,k)y^jz^k\left[\frac{G_2(G_1,z)^j}{y^j}-\frac{G_2(G_1,z)^{k_2}}{y^{k_2}}\right]\\
&-\phi_0 V(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}}-\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \qquad (23)\\
&+\phi_0\sum_{k=1}^{k_3-1}v(0,0,k)z^k\left[\frac{G_3(G_1,y)^k}{z^k}-\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]+\phi_0 V(G_1,y,z)-\phi_0\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}.
\end{aligned}
$$

For $k_2, k_3 \geq 1$, the above functional equation is rearranged as follows: Setting $y=\delta_r(z)$, we have

$$
\begin{aligned}
&\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}\phi(0,j,k)y^jz^k\left[1-\frac{G_2(G_1,z)^j}{y^j}\right]_{y=\delta_r(z)}\\
&\quad+\Phi(0,0,z)\left[-\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)}+\sum_{k=1}^{k_3-1}\phi(0,0,k)z^k\left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}-\frac{G_3(G_1,y)^k}{z^k}\right]_{y=\delta_r(z)}\\
&\quad=\phi_0\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}v(0,j,k)y^jz^k\left[\frac{G_2(G_1,z)^j}{y^j}-1\right]_{y=\delta_r(z)}-\phi_0 V(0,0,z)\left[1-\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)}\\
&\qquad+\phi_0\sum_{k=1}^{k_3-1}v(0,0,k)z^k\left[\frac{G_3(G_1,y)^k}{z^k}-\frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)}\\
&\qquad+\phi_0\left[V(G_1,y,z)-\frac{G_3(G_1(y,z),y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)}.
\end{aligned}
$$

In succession to k_2, we consider the zeros $z=\omega_s$ for k_3, i.e. $z=\omega_s,\ s=1,2,\cdots,k_3-1$ and $\omega_{k_3}=1$ such as

$$
\left[1-\frac{G_3\{G_1(y,z),z\}^{k_3}}{y^{k_3}}\right]_{y=\delta_r(z)}=0. \qquad (40)
$$

Setting $z=\omega_s$, we have

$$
\begin{aligned}
&\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}\phi(0,j,k)(\delta_r(\omega_s))^j(\omega_s)^k\left[1-\frac{G_2(G_1(y,\omega_s),\omega_s)^j}{y^j}\right]_{y=\delta_r(\omega_s)}\\
&\quad+\sum_{k=1}^{k_3-1}\phi(0,0,k)(\omega_s)^k\left[1-\frac{G_3(G_1(y,\omega_s),y)^k}{\omega_s^k}\right]_{y=\delta_r(\omega_s)}\\
&\quad=\phi_0\sum_{j=1}^{k_2-1}\sum_{k=0}^{\infty}v(0,j,k)(\delta_r(\omega_s))^j(\omega_s)^k\left[\frac{G_2(G_1(y,\omega_s),\omega_s)^j}{y^j}-1\right]_{y=\delta_r(\omega_s)} \qquad (41a)\\
&\quad+\phi_0\sum_{k=1}^{k_3-1}v(0,0,k)(\omega_s)^k\left[\frac{G_3(G_1(y,\omega_s),y)^k}{\omega_s^k}-1\right]_{y=\delta_r(\omega_s)}\\
&\quad+\phi_0\left[V(G_1(y,\omega_s),y,\omega_s)-1\right]_{y=\delta_r(\omega_s)}.
\end{aligned}
$$

For $k_0 :=^{\forall} k \in \{0, 1, 2 \cdots < \infty\}$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{k_2-1} \phi(0, j, k_0)(\delta_r(\omega_s))^j (\omega_s)^{k_0} \left[1 - \frac{G_2(G_1(y, \omega_s), \omega_s)^j}{y^j}\right]_{y=\delta_r(\omega_s)} \\
&\quad + \sum_{k=1}^{k_3-1} \phi(0, 0, k)(\omega_s)^k \left[1 - \frac{G_3(G_1(y, \omega_s), y)^k}{\omega_s^k}\right]_{y=\delta_r(\omega_s)} \\
&= \phi_0 \sum_{j=1}^{k_2-1} v(0, j, k_0)(\delta_r(\omega_s))^j (\omega_s)^{k_0} \left[\frac{G_2(G_1(y, \omega_s), \omega_s)^j}{y^j} - 1\right]_{y=\delta_r(\omega_s)} \\
&\quad + \phi_0 \sum_{k=1}^{k_3-1} v(0, 0, k)(\omega_s)^k \left[\frac{G_3(G_1(y, \omega_s), y)^k}{\omega_s^k} - 1\right]_{y=\delta_r(\omega_s)} \\
&\quad + \phi_0 \left[V(G_1(y, \omega_s), y, \omega_s) - 1\right]_{y=\delta_r(\omega_s)} .
\end{aligned} \tag{41b}$$

Here, we introduce,

$$\alpha\{j, r, s\}_{k_0} := (\delta_r(\omega_s))^j (\omega_s)^{k_0} \left[1 - \frac{G_2(G_1(y, \omega_s), \omega_s)^j}{y^j}\right]_{y=\delta_r(\omega_s)}, \text{ and}$$

$$\alpha\{k, r, s\} := (\omega_s)^k \left[1 - \frac{G_3(G_1(y, \omega_s), y)^k}{\omega_s^k}\right]_{y=\delta_r(\omega_s)} .$$

Then, we obtain the following simultaneous linear equation for $\{\phi(0, j, k_0)\}$ and $\{\phi(0, 0, k)\}$, which are guaranteed the existence and the uniqueness by $\rho < 1$. (j, k are independent variables):

$$\begin{aligned}
&\sum_{j=1}^{k_2-1} \phi(0, j, k_0)\alpha\{j, r, s\}_{k_0} + \sum_{k=1}^{k_3-1} \phi(0, 0, k)\alpha(k, r, s) \\
&= \phi_0 \sum_{j=1}^{k_2-1} v(0, j, k_0)\alpha_0\{j, r, s\}_{k_0} + \phi_0 \sum_{k=1}^{k_3-1} v(0, 0, k)\alpha_0(k, r, s) \\
&\quad + \phi_0 \left[V(G_1(y, \omega_s), y, \omega_s) - 1\right]_{y=\delta_r(\omega_s)}, \\
&\qquad r = 1, 2, \cdots, k_2 - 1, \qquad s = 1, 2, \cdots, k_3 - 1.
\end{aligned} \tag{42}$$

This linear equation has the square matrix $(k_2 - 1 + k_3 - 1) \times (k_2 - 1 + k_3 - 1)$. Note that coefficients $\alpha\{j, r, s\}_{k_0}, \alpha(k, r, s)$ and $v(0, j, k_0) \cdot \alpha_0\{j, r, s\}_{k_0}, v(0, 0, k) \cdot \alpha_0(k, r, s)$ are the known functions, which are uniquely arranged in the matrix, where $1 \leq j, r \leq k_2 - 1$ and $1 \leq k, s \leq k_3 - 1$, $k_0 =^{\forall} k \in \{0, 1, 2, \cdots < \infty\}$, ($\phi_0$ is still unknown).

$$\therefore \quad \phi(0, j, k_0) := \phi_0 \Delta_j, \quad \Delta_j := \frac{|D_j|}{|D|}, \quad j = 1, 2, \cdots, k_2 - 1, \tag{43a}$$

$$\phi(0, 0, k) := \phi_0 \Delta^k, \quad \Delta^k := \frac{|D_{k_2-1+k}|}{|D|}, \quad k = 1, 2, \cdots, k_3 - 1. \tag{43b}$$

Using $y = \delta_r(z)$, we obtain $\widehat{\Phi}(0, 0, z)$ given by

$$\begin{aligned}
&\widehat{\Phi}(0, 0, z) \left[1 - \frac{G_3(G_1, y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)} + \phi_0 \sum_{j=1}^{k_2-1} \Delta_j (\delta_r(z))^j (\omega_s)^{k_0} \left[1 - \frac{G_2(G_1, z)^j}{y^j}\right]_{y=\delta_r(z)} \\
&\quad + \phi_0 \sum_{k=1}^{k_3-1} \Delta^k \left[\frac{G_3(G_1, y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1, y)^k}{z^k}\right]_{y=\delta_r(z)}
\end{aligned}$$

$$
\begin{aligned}
=&\phi_0 \sum_{j=1}^{k_2-1} v(0,j,k_0)(\delta_r(z))^j z^{k_0} \left[\frac{G_2(G_1,z)^j}{y^j} - 1\right]_{y=\delta_r(z)} \\
&- \phi_0 V(0,0,z)\left[1 - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)} \\
&+ \phi_0 \sum_{k=1}^{k_3-1} v(0,0,k) z^k \left[\frac{G_3(G_1,y)^k}{z^k} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)} \\
&+ \phi_0 \left[V(G_1,y,z) - \frac{G_3(G_1(y,z),y)^{k_3}}{z^{k_3}}\right]_{y=\delta_r(z)} .
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
&\widehat{\Phi}(0,y,z)\left[1 - \frac{G_2\{G_1(y,z),z\}^{k_2}}{y^{k_2}}\right] + \phi_0 \sum_{j=1}^{k_2-1} \Delta_j (\delta_r(z))^j (\omega_s)^{k_0} \left[1 - \frac{G_2(G_1,z)^j}{y^j}\right]_{y=\delta_r(z)} \\
&+\widehat{\Phi}(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \phi_0 \sum_{k=1}^{k_3-1} \Delta^k \left[\frac{G_3(G_1,y)^{k_3}}{z^{k_3}} - \frac{G_3(G_1,y)^k}{z^k}\right]_{y=\delta_r(z)} \\
&= - \phi_0 V(0,y,z))\left[1 - \frac{G_2\{G_1,z\}^{k_2}}{y^{k_2}}\right] + \phi_0 \sum_{j=1}^{k_2-1} v(0,j,k_0) y^j z^{k_0} \left[\frac{G_2(G_1,z)^j}{y^j} - \frac{G_2(G_1,z)^{k_2}}{y^{k_2}}\right] \\
&\quad - \phi_0 V(0,0,z)\left[\frac{G_2(G_1,z)^{k_2}}{y^{k_2}} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] \\
&\quad + \phi_0 \sum_{k=1}^{k_3-1} v(0,0,k) z^k \left[\frac{G_3(G_1,y)^k}{z^k} - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] + \phi_0 \left[V(G_1,y,z) - \frac{G_3(G_1,y)^{k_3}}{z^{k_3}}\right] .
\end{aligned} \tag{45}
$$

In the above equation, $\Phi(0,0,z)$ denoted by $\widehat{\Phi}(0,0,z)$ has been explicitly determined as in (44). Denoting the above determined $\Phi(0,0,z)$ by $\widehat{\Phi}(0,0,z)$, we also obtain $\widehat{\Phi}(0,y,z)$ as in (45). (That is, Step 1 has been completed).

Step 2:

$$
\begin{aligned}
\Phi_1(0,y,z) =&\phi_0\{V(G_1(y,z),y,z) - V(0,y,z)\} \\
&+ \Phi_2\{G_1(y,z),0,z\} - \Phi_2(0,0,z) + \Phi_3\{G_1(y,z),y,0\} - \Phi_3(0,y,0),
\end{aligned} \tag{22a}
$$

where $\Phi_2(x,y,z)$ and $\Phi_3(x,y,z)$ are given by Eqs. (22b), (22c), respectively.

Step 3: Using $\Phi_2(x,y,z), \Phi_3(x,y,z)$ and $\widehat{\Phi}(0,y,z)$, we can express $P_n(x,y,z), n = 1,2,3$, by which we can obtain $P_n(\cdot), \Pi_n(\cdot)$ and $W_n^*(s), n = 1,2,3$.

Therefore, from the above analyses, the final result for the case of ($k_1 = \infty, k_2, k_3 \geq 1$) is summarized as follows:

Theorem 3.3. For the case of ($k_1 = \infty, k_2, k_3 \geq 1$), the LSTs $W_n^*(s), n = 1,2,3$ are given by the top-down procedure, where $\Phi(0,0,z)$ and $\Phi(0,y,z)$ are given by (44) and (45), respectively. $P_1(x,y,z), P_2(0,y,z)$ and $P_3(0,0,z)$ are given by (24a), (24b) and (24c), respectively. □

4 Complements (Generalization)

We introduce the following notation for $n = 1, 2, \cdots, N$:

$$r_n := \lambda_n/\lambda, \qquad \lambda := \sum_{n=1}^{N} \lambda_n, \qquad \lambda_n^+ := \sum_{i=1}^{n} \lambda_i,$$
$$\rho_n := \lambda_n h_n, \qquad \rho := \sum_{n=1}^{N} \rho_n, \qquad \rho_n^+ := \sum_{i=1}^{n} \rho_i, \tag{47a}$$
$$h := \sum_{n=1}^{N} r_n h_n, \qquad h^{(2)} := \sum_{n=1}^{N} r_n h_n^{(2)}$$

and

$$Q_n(x) := H_n^*\{\lambda_n(1-x)\}, \quad n = 1, 2, \cdots, N,$$
$$H_n^+(s) := \frac{1}{\lambda_n^+}\sum_{i=1}^{n} \lambda_i H_i^*(s), \qquad h_n^+ := \frac{\rho_n^+}{\lambda_n^+}, \qquad h_n^{+(2)} := \frac{1}{\lambda_n^+}\sum_{i=1}^{n} \lambda_i h_i^{(2)}, \tag{47b}$$
$$\Theta_n^*(s) = H_n^*\{s + \lambda_n(1-\Theta_n^*(s))\},$$
$$E(\Theta_n) = \frac{h_n}{1-\rho_n}, \quad E(\Theta_n^2) = \frac{h_n^{(2)}}{(1-\rho_n)^3},$$
$$\Theta_{n-1}^+(s) = H_{n-1}^+\{s + \lambda_{n-1}^+(1-\Theta_{n-1}^+(s))\}, \tag{47c}$$
$$Q_v(0,0,\cdots,0) = 0, \ \pi_0 := \sum_{n=1}^{N} \Pi_n(0,0,\cdots,0).$$

Additionally, we introduce GFs $\Pi_n(z_1, z_2, \cdots, z_N)$, $n = 1, 2, \cdots, N$ for queue-length distributions defined at message departure epochs, which is an extension of $\Pi_n(x, y, z)$ in Section 3.

4.1 Exhaustive Service Model

In Katayama and Kobayashi [12] and Katayama [13], the time-limited service priority model with the maximum-attendance-time $(T_1, T_2, \cdots, T_N)$ was analyzed in 2007, i.e. the exhaustive service priority model with ($k_1 = k_2 = \cdot\cdot = k_N = \infty$) is equivalent to $(T_1 = T_2 = \cdot\cdot = T_N = \infty)$. The outline of contents is summarized in this section.

4.1.1 Basic Invariant Relation

We know from the derivation of Eqs. (34a)∼(34c) in Section 3 that the right-hand side of (48a) depends on only $z_n, z_{n+1} \cdots, z_N$. That is, the key function φ_n is defined as follows:

$$\Pi_n(z_1, z_2, \cdots, z_N)\{z_n - Q_n\}/Q_n + \Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, z_{n+2}, \cdots, z_N)$$
$$=: \varphi_n(z_n, z_{n+1}, \cdots, z_N). \tag{48a}$$

The zero, $z_n = \bar{\delta}_n$ of $(z_n - Q_n = 0)$ can be expressed as $\bar{\delta}_n(z_1, z_2, \cdots, z_{n-1}, z_{n+1}, \cdots, z_N)$. For brevity, we use $\bar{\delta}_n \equiv \bar{\delta}_n(z_1, z_2, \cdots, z_{n-1}, z_{n+1}, \cdots, z_N)$. Using the relation,

$$\Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, z_{n+2}, \cdots, z_N) = \varphi_n(\bar{\delta}_n, z_{n+1}, \cdots, z_N),$$

we have,

$$\Pi_n(z_1, z_2, \cdots, z_N)\{z_n - Q_n\}/Q_n$$
$$= \varphi_n(z_n, z_{n+1}, \cdots, z_N) - \varphi_n(\bar{\delta}_n, z_{n+1}, \cdots, z_N). \tag{48b}$$

Adding both sides of (48b) over $n = 1, 2, \cdots, N$, we obtain finally the following invariant relation:

Lemma 4.1.

$$\sum_{n=1}^{N} \frac{\Pi_n(z_1, z_2, \cdots, z_N)\{z_n - Q_n\}}{Q_n(z_1, z_2, \cdots, z_N)} = \pi_0\{Q_v(z_1, z_2, \cdots, z_N) - 1\}. \tag{48c}$$

□

Here, "invariant relation" means that the both-sides of Eq. (48c) don't contain any term concernig with the controllable parameters k_n and $T_n, n = 1, 2, \cdots, N$. (For $0 \leq T_n \leq \infty$, it is not appeared at the both-sides in Katayama [13]).

Eq. (48a) can be rewritten as

$$\begin{aligned}\Pi_n^+&(z_1, z_2, \cdots, z_N)\\ =&\{\Pi_n(z_1, z_2, \cdots, z_N) - \Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, z_{n+2}, \cdots, z_N)\}\\ &+ \varphi_n(z_n, z_{n+1}, \cdots, z_N),\end{aligned} \tag{48d}$$

where $\Pi_n^+(z_1, z_2, \cdots, z_N) := \Pi_n(z_1, z_2, \cdots, z_N)z_n/Q_n$, which represents the GF of the number of messages in class-1, class-2,$\cdots$, class-N at a <service-beginning> epoch of service time $H_n, n = 1, 2, \cdots, N$. The two terms on the right-hand side represent the GF of the number of messages at service beginning epoch of the service time $H_n, n = 1, 2, \cdots, N$. In succession, a single server remains at S_n for the exhaustive service for class-n messages. The third term $\varphi_n(z_n, z_{n+1}, \cdots, z_N)$ represents the GF of the number of messages of class-n, class-$(n+1)$, $\cdots$, class-N at service beginning epoch, where the moving single server moves from the other classes. The movement of the single server is described as, $S_n \leftarrow S_{n+1}, S_{n+2}, \cdots, S_N$. Indeed, there is no message of class-1, class-2, $\cdots$, class-$(n-1)$ in the queueing system as shown by the GF of $\varphi_n(z_n, z_{n+1}, \cdots, z_N)$. Similarly, the first part is described as $S_n \rightarrow S_n$.

We sum for the both-sides of (48d):

$$\begin{aligned}\sum_{n=1}^{N}\Pi_n^+(z_1, z_2, \cdots, z_N) =& \sum_{n=1}^{N}\{\Pi_n(z_1, z_2, \cdots, z_N) - \Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, \cdots, z_N)\}\\ &+ \sum_{n=1}^{N}\varphi_n(z_n, z_{n+1}, \cdots, z_N)\\ =& - \sum_{n=1}^{N}\Pi_n(0, 0 \cdots, 0) + \pi_0\{Q_v(z_1, z_2, \cdots, z_N) - Q_v(0, 0, \cdots, 0)\}\\ =& - \pi_0 + \pi_0 Q_v(z_1, z_2, \cdots, z_N) \qquad (Q_v(0, 0, \cdots, 0) = 0 \ \because \text{MV rule})\\ =&\pi_0\{Q_v(z_1, z_2, \cdots, z_N) - 1\}.\end{aligned} \tag{48e}$$

$$\therefore \quad \sum_{n=1}^{N}\Pi_n^+(z_1, z_2, \cdots, z_N) = \pi_0\{Q_v(z_1, z_2, \cdots, z_N) - 1\}, \tag{48f}$$

which leads to Lemma 4.1.

Further, using (48b), Eq. (48f) can be rearranged as follows:

$$\left[\sum_{n=1}^{N}\{\Pi_n(z_1, z_2, \cdots, z_N) - \Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, \cdots, z_N)\} - \sum_{n=1}^{N}\Pi_n(0, 0, \cdots, 0)\right]$$

$$+ \pi_0 Q_v(z_1, z_2, \cdots, z_N) = \sum_{n=1}^{N}\Pi_n^+(z_1, z_2, \cdots, z_N). \tag{48g}$$

This is a balance equation, for which a probabilistic meaning is given as follows. The first two terms in $[\sim]$ on the left-hand side of (48g) represents the GF of the number of messages at a service-beginning epoch after service time H_n and in succession, a single server continues serving of class-n messages because of the exhaustive service discipline, which is represented by

$$-\Pi_n(z_1, z_2, \cdots, z_{n-1}, 0, z_{n+1}, \cdots, z_N).$$

The fouth term $\pi_0 Q_v(z_1, z_2, \cdots, z_N)$ represents the GF of the number of messages at service-beginning epochs after a vacation time V during the last vacation period (MV). On the other hand, the GF of the right-hand side, $\sum_{n=1}^{N} \Pi_n^+(z_1, z_2, \cdots, z_N)$ represents the GF of the number of messages for all classes at service-beginning epochs in the queueing system, which consists of the former exclusive three terms on the left-hand side. Here, $-\sum_{n=1}^{N} \Pi_n(0, 0, \cdots, 0)$ is the remainder term, by which it is guaranteed that the GF of the number of messages in $[\sim]$ represents the GF of the number of messages at <service-beginning> epochs of the service time $H_n, n = 1, 2, \cdots, N$. That is, it is guaranteed by the term $(-\pi_0)$ that there is at least one message in the queueing system.

Next, in order to derive functional equations for $\varphi_n(z_n, z_{n+1}, \cdots, z_N)$, $n = 1, 2, \cdots, N$, we introduce a set of $\{x_{k,j}; j = 1, 2, \cdots, k-1\}$ satisfying the following simultaneous equations,

$$x_{k,j} = Q_j\{x_{k,1}, x_{k,2}, \cdots, x_{k,k-1}, z_k, z_{k+1}, \cdots, z_N\} \text{ for } j = 1, 2, \cdots, k-1, k = 2, 3, \cdots, N.$$

The element $x_{k,j}$ is thus a function of $z_k, z_{k+1}, \cdots, z_N$ denoted by

$$x_{k,j} := x_{k,j}(z_k, z_{k+1}, \cdots, z_N).$$

Further, we also introduce two functions defined by

$$x_k(z_k, z_{k+1}, \cdots, z_N) := \sum_{j=1}^{k-1} \lambda_j x_{k,j}(z_k, z_{k+1}, \cdots, z_N),$$

$$x_k(z) := x_k(z, 1, \cdots, 1).$$

Then, we obtain the following lemma on $x_k(z), k \geq 2$. Eqs. (49) and (49a) have one and only one zero $x_k(z)$ in the unit circle $|x_k(z)/\lambda_{k-1}^+| \leq 1$:

Lemma 4.2.

$$x_k(z) - \sum_{j=1}^{k-1} \lambda_j H_j^*\{\lambda_k^+ - \lambda_k z - x_k(z)\} = 0, \quad k = 2, 3, \cdots, N, \tag{49}$$

$$\frac{x_k(z)}{\lambda_{k-1}^+} = \frac{1}{\lambda_{k-1}^+} \sum_{j=1}^{k-1} \lambda_j H_j^* \left\{ \lambda_k(1-z) + \lambda_{k-1}^+ \left[1 - \frac{x_k(z)}{\lambda_{k-1}^+} \right] \right\}. \tag{49a}$$

Further, inserting $x_k(1 - s/\lambda_k)/\lambda_{k-1}^+ = \Theta_{k-1}^+(s)$ to both sides of (49a), it follows that

$$\Theta_{k-1}^+(s) = H_{k-1}^+\{s + \lambda_{k-1}^+(1 - \Theta_{k-1}^+(s))\}. \tag{49b}$$

Note that $\Theta_{k-1}^+(s)$ is the LST of the DF of the busy period in the standard $M/G/1$ queue with $\lambda := \lambda_{k-1}^+$ and $H^*(s) := H_{k-1}^+(s)$. That is, we have

$$E(\Theta_{k-1}^+) = \frac{h_{k-1}^+}{1 - \rho_{k-1}^+}, \quad E(\Theta_{k-1}^{+2}) = \frac{h_{k-1}^{+(2)}}{(1 - \rho_{k-1}^+)^3}. \tag{49c}$$

□

4.1.2 Functional Equation for $\varphi_n(z_n, z_{n+1}, \cdots, z_N)$

Using $\beta_k \equiv \beta_k(z) := \lambda_{k-1}^+ - x_k(z)$ and the set $\{x_{k,j}, k \geq 2\}$, we obtain from Lemma 4.1 that

$$\begin{aligned}
&\varphi_n(z_n, z_{n+1}, \cdots, z_N) - \varphi_n\{\beta_n, z_{n+1}, \cdots, z_N\} \\
&+ \sum_{j=n+1}^{N} \{\varphi_j(z_j, z_{j+1}, \cdots, z_N) - \varphi_j(\beta_j, z_{j+1}, \cdots, z_N)\} \\
&= \pi_0\{Q_v(x_{n,1}, x_{n,2}, \cdots, x_{n,n-1}, z_n, z_{n+1}, \cdots, z_N) - 1\}.
\end{aligned} \tag{50a}$$

(Note that β_k instead of $\bar{\delta}_k$ does not contain the variables $z_1, z_2, \cdots, z_{k-1}$).
Setting $z_n = z, z_{n+1} = \cdots = z_N = 1$, we obtain the following functional equation of $G_n(z)$:

$$G_n(z) := \varphi_n(z, 1, 1, \cdots, 1), \tag{51}$$

where

$$G_k(z) - G_k\{f_k(z)\} = g_k(z), \qquad k = n, n+1, \cdots, N, \tag{51a}$$

$$f_k(z) := \lambda_{k-1}^+ - x_k(z), \quad \because\ \beta_k^k(z) = \sum_{j=1}^{k-1} \lambda_j(1 - x_{k,j}), \tag{51b}$$

$$g_k(z) := \sum_{j=k+1}^{N} \{G_j(\beta_j^k(z)) - G_j(1)\} + \pi_0\{V^*(\alpha_k(z)) - 1\}, \tag{51c}$$

$$\alpha_k(z) := \lambda_{k-1}^+ - x_k(z) + \lambda_k(1 - z), \quad \text{(see Lemma 4.2)}, \tag{51d}$$

$$\begin{aligned}
\beta_n^k(z) :=& - \lambda_n(1 - z_n) + \sum_{j=1}^{k-1} \lambda_j(1 - x_{n,j}) + \lambda_k(1 - z) + \sum_{j=k+1}^{N} \lambda_j(1 - z_j) \qquad (51e) \\
=& \sum_{j=1}^{k-1} \lambda_j(1 - x_{k,j}) + \lambda_k(1 - z) + \sum_{\substack{j=k+1 \\ (j \neq n)}}^{N} \lambda_j(1 - z_j).
\end{aligned}$$

$$\begin{aligned}
\Pi_n(z) :=& \frac{\Pi_n(1, 1, \cdots, z_n = z, 1, \cdots, 1)}{\Pi_n(1, 1, \cdots, 1)} \\
=& \frac{Q_n(z)/r_n}{z - Q_n(z)}\{G_n(z) - G_n(1)\}.
\end{aligned} \tag{52}$$

Remark 4.1. From the result of $\Pi_n(z) = \sum_{i=1}^{\infty} p_n(i)Q_n(z; i)$ and Eq. (52), we obtain that $G_n(z) - G_n(1) = P_n(z) - P_n(1)$, which corresponds to $\varphi_n(z, 1, \cdots, 1) = G_n(z) = P_n(z), n = 1, 2, 3.$ Two GFs $G_n(z)$ and $P_n(z)$ represent GFs defined at <service-beginning epochs>.

4.1.3 Waiting Time Analysis

For $k_1 = k_2 = \cdots = k_N = K = \infty$, Theorem 4.1 provides the LST of waiting time distribution for class n, whereas the mean waiting time is given in Corollary 4.1.

Theorem 4.1.

$$W_n^*(s) = \frac{(1 - \rho_n)s}{s - \lambda_n\{1 - H_n^*(s)\}} \frac{\{G_n(1 - s/\lambda_n) - G_n(1)\}}{-s/\lambda_n}. \tag{53}$$

□

Corollary 4.1.

$$E(W_n)_{K=\infty} = \frac{\lambda_n h_n^{(2)}}{2(1 - \rho_n)} + \frac{G_n^{(2)}(1)}{2r_n\lambda_n(1 - \rho_n)}, \quad n = 1, 2, \cdots, N. \tag{53a}$$

In particular, for $n = N$,

$$E(W_N)_{K=\infty} = \frac{\lambda_N h_N^{(2)}}{2(1-\rho_N)} + \frac{G_N^{(2)}(1)}{2\lambda_N(1-\rho_N)},$$
$$G_N^{(2)}(1) = \frac{\lambda_N \lambda_{N-1}^{+}(1-\rho_N)^2 h_{N-1}^{+(2)} + \lambda_N^2(\rho_{N-1}^{+})^2 h_N^{(2)}}{(1-\rho)(1-\rho+2\rho_{N-1}^{+}\rho_N)} + \frac{2\lambda_N(1-\rho)(1-\rho_N)^2}{(1-\rho)(1-\rho+2\rho_{N-1}^{+}\rho_N)}\frac{v^{(2)}}{2v}. \tag{53b}$$

A set of recursive equations to calculate $G_k^{(2)}(1), k = n, n+1, \cdots, N$ in (53a) is given by

$$(1-\rho_k^{+})(1-\rho_k^{+}+2\rho_k\rho_{k-1}^{+})G_k^{(2)}(1) = r_k\lambda_k\lambda_{k-1}^{+}(1-\rho_k)^2 h_{k-1}^{+(2)} + r_k\lambda_k^2(\rho_{k-1}^{+})^2 h_k^{(2)} + \lambda_k^2(1-\rho_k)^2 \sum_{j=k+1}^{N}\left\{\frac{r_j h_j^{(2)}}{(1-\rho_j)^2} + \frac{h_j^2}{(1-\rho_j)^2}G_j^{(2)}(1)\right\} + 2r_k\lambda_k(1-\rho)(1-\rho_k)^2\frac{v^{(2)}}{2v}. \tag{53c}$$

□

Remark 4.2. From the functional equation analysis with Eqs. (51a)∼ (51c), we obtain

$$G_n(z) - G_n\{f_n(z)\} = \pi_0\{V^*(\alpha_n(z)) - 1\} + \sum_{j=n+1}^{N}\{G_j(\beta_j^k(z)) - G_j(1)\}, \tag{54}$$

which enables us to derive the recursive formula (53c) of the moments of waiting time. (The descent-order starting from $k = N$ is necessary for getting of $G_n^{(2)}(1)$ in (53c) and $G_n(z)$ in (54)).

Remark 4.3. The functional relations on $\Phi_n(x, y, z)$ for $N = 3$ are given by:

$$\Phi_1(0, y, z) = \phi_0\{V(G_1(y, z), y, z) - V(0, y, z)\} + \Phi_2(G_1(y, z), 0, z) - \Phi_2(0, 0, z) + \Phi_3(G_1(y, z), y, 0) - \Phi_3(0, y, 0), \tag{55a}$$
$$\Phi_2(x, 0, z) = \phi_0\{V(0, G_2(x, z), z) - V(0, 0, z)\} + \Phi_1(0, G_2(x, z), z) - \Phi_1(0, 0, z) + \Phi_3(0, G_2(x, z), 0) - \Phi_3(0, 0, 0), \tag{55b}$$
$$\Phi_3(x, y, 0) = \phi_0\{V(0, 0, G_3(x, y)) - V(0, 0, 0)\} + \textstyle\sum_{n=1}^{2}\{\Phi_n(0, 0, G_3(x, y)) - \Phi_n(0, 0, 0)\}. \tag{55c}$$

Further, by use of only above functional relations (i.e. no use of $\Pi_n(x, y, z)$), such iterative functional equation may be derived, as corresponding to Eq.(16d) for the case of ($k_1 = k_2 = \infty$), or not. It is unnatural in the top-down procedure to introduce such inconsistent $\Pi_n(x, y, z)$ at the first in Section 3. It follows from using Lemma 4.1and the set $\{x_{k,j}\}$ introduced before Lemma 4.2 that both iterative functional equations (51a) and (54) have been derived, which are considered as modified functional equations of (16d).

4.2 Semi-Exhaustive Service Model

4.2.1 Delay Cycle Analysis

For the case of ($k_1 = k_2 = \cdots = k_N = 1$), the delay cycle method is effective. Applying the delay cycle analysis and introducing a super-message composed of messages served in a busy period, we derive explicitly the LST and the first moment of the message waiting time distribution for each class in the M/G/1 priority queues with multiple vacation. Note that a super-message of class n enters service only when there is no message of class $i, i = 1, 2, \cdots, n-1$ in the queueing system. The system is either in a delay cycle, called a V-cycle, initiated by a vacation time V with probability P_V, or in a delay cycle, called a B_i-cycle, initiated by a service time of a super-message of class $i, B_i \equiv \Theta_i$ with probability $P_i, i = n+1, n+2, \cdots, N$. Note that for the derivation of $W_n^*(s)$, the server utilization for V-cycle and B_i-cycle are assigned to $1-\rho$ and ρ_i, respectively, that is, $P_i/P_V = \rho_i/(1-\rho)$. While the possible server-utilization is $1-\rho_n^+ \ (= P_V + P_i)$. Then, we have the following results.

Theorem 4.2.

$$W_n^*(s) = \frac{1-\rho}{1-\rho_n^+} W_n^*(s|V\text{-cycle}) + \sum_{i=n+1}^{N} \frac{\rho_i}{1-\rho_n^+} W_n^*(s\,|B_i\text{-cycle}), \tag{56a}$$

$$W_n^*(s|V\text{-cycle}) := \frac{1-V^*(\sigma_{n-1}(s))}{sv/(1-\rho_{n-1}^+)} U_n^*(s), \quad W_n^*(s|B_i\text{-cycle}) = \frac{1-\Theta_i^*(\sigma_{n-1}(s))}{sE(\Theta_i)/(1-\rho_{n-1}^+)} U_n^*(s),$$

$$U_n^*(s) := W_n^*(s)_{M/D/1} \times W_n^*(s)_{M/H/1}, \tag{56b}$$

$$W_n^*(s)_{M/D/1} := \frac{s(1-\lambda_n E(D_n))}{s-\lambda_n+\lambda_n D_n^*(s)}, \quad W_n^*(s)_{M/H/1} := \frac{s(1-\lambda_n E(H_n))}{s-\lambda_n+\lambda_n H_n^*(s)},$$

$$D_n^*(s) := B_n^*[\lambda_{n-1}^+\{1-\Theta_{n-1}^+(s)\}] = \Theta_n^*(\sigma_{n-1}(s)-s) \neq \Theta_n^*(\sigma_{n-1}(s)), \tag{56c}$$

$$\sigma_{n-1}(s) := s + \lambda_{n-1}^+ - \lambda_{n-1}^+ \Theta_{n-1}^+(s).$$

Here, the service time D_n is also called **the interception time** generated by messages of class-1, class-2, $\cdots$, class-(n-1). The LST $D_n^*(s)$ in (56c) means the increment of the waiting time of class-n messages caused by the interception of higher priority messages. In the case of $D_n = 0$, only the second factor in (56b), $W_n^*(s)_{M/H/1}$, contributes to the LST of the DF for the waiting time of class-n messages that arrive during V-cycle and B_i-cycle. It should be noted that $\Theta_n^*(\sigma_{n-1}(s)-s) \neq \Theta_n^*(\sigma_{n-1}(s))$ in (56c). That is, the LST $D_n^*(s)$ of the DF of <the service time D_n> does not correspond to $\Theta_n^*(\sigma_{n-1}(s)) \neq D_n^*(s)$, but to $H_n^*(s)$, i.e. the same category with the LST of the DF of <the service time H_n> for class-n messages, see details of Katayama [10]. □

For $N = 2, n = 2$ and $k_2 = 1$ (the semi-exhaustive service), it follows from Eqs. (8c) and (11) in Section 2 that

$$\Pi_2(y) = \frac{C_2\phi_0 Q_2(y)}{y-Q_2(y)}(1-y^{-1})\{\Phi(y)-1+V(0,y)\}, \tag{57}$$

where

$$\Phi(y) = 1-V(0,y)+\frac{y\{V(G_1(y),y)-1\}}{y-G_2(G_1(y))}.$$

$$\therefore \quad \Pi_2(y) = \frac{C_2\phi_0 Q_2(y)}{y-Q_2(y)}(y-1)\cdot\frac{V(G_1(y),y)-1}{y-G_2(G_1(y))}. \tag{57a}$$

Therefore, setting $y = 1 - s/\lambda_2$, we have

$$W_2^*(s) = \Pi_2(1-s/\lambda_2)/H_2^*(s), \qquad \text{(see } W_2^*(s) \text{ in Theorem 2.2).} \tag{58}$$

It should be pointed out that $W_n^*(s|V\text{-cycle})$ in Theorem 4.2 has been analogically derived by using the result of minimum N = 2, i.e. $W_2^*(s)$ in (58), (see Katayama [9, 10] on $W_2^*(s)$).

Corollary 4.2. For $k_1 = k_2 = \cdots = k_N = 1 = K$, the mean waiting time for class-n is given by:

$$E(W_n)_{K=1} = E(W_n)_0 + \frac{1-\rho}{(1-\rho_{n-1}^+)(1-\rho_n^+)}\frac{v^{(2)}}{2v}, \quad n = 1, 2, \cdots, N, \tag{59}$$

where

$$E(W_n)_0 := \frac{\lambda_n h_n^{(2)}}{2(1-\rho_n)}\left[1+\frac{(\rho_{n-1}^+)^2}{(1-\rho_{n-1}^+)(1-\rho_n^+)(1-\rho_n)}\right] \tag{59a}$$
$$+\frac{1}{2(1-\rho_{n-1}^+)(1-\rho_n^+)}\left[\sum_{i=1}^{n-1}\lambda_i h_i^{(2)}+\sum_{i=n+1}^{N}\frac{\lambda_i h_i^{(2)}}{(1-\rho_i)^2}\right].$$

□

Remark 4.4. The structure of T-cycle

For the analysis for $W_n^*(s)$, T-cycle is defined as the delay cycle initiated with an initial delay T and generated by messages of class 1, 2, $\cdots$, n, where $T^*(s)$ is the LST of the DF for the initial delay T. A period θ of T-cycle is given by

$$\theta^*(s) := T^*\{\sigma_{n-1}(s)\},$$
$$\sigma_{n-1}(s) := s+\lambda_{n-1}^+ - \lambda_{n-1}^+\Theta_{n-1}^+(s),$$

where $\Theta_{n-1}^+(s)$ is the LST of the DF for the length Θ_{n-1}^+ of a busy period generated by messages of class $1, 2, \cdots, n-1$, which is a solution to the equation

$$\Theta_{n-1}^+(s) = H_{n-1}^+\{s+\lambda_{n-1}^+ - \lambda_{n-1}^+\Theta_{n-1}^+(s)\}. \tag{60}$$

If we denote by $W_n^*(s|T$-cycle), the LST of the DF for the waiting time of a message of class n that arrives during a T-cycle generated by an initial delay T is expressed by a product form:

$$W_n^*(s|\text{T-cycle}) = \frac{1-\theta^*(s)}{s\text{E}(\theta)}\Phi_n^*(s) = \frac{1-T^*\{\sigma_{n-1}(s)\}}{s\text{E(T)}/(1-\rho_{n-1}^+)}\Phi_n^*(s), \tag{61}$$

where $\Phi_n^*(s)$ is given by the following form using the Pollaczek-Khintchine formula,

$$\Phi_n^*(s) := \frac{s(1-\lambda_n E(C_n))}{s-\lambda_n+\lambda_n C_n^*(s)}, \tag{62}$$

where $C_n^*(s)$ is the LST for the completion time C_n defined by $C_n^*(s) := H_n^*\{\sigma_{n-1}(s)\}$.

5 Concluding Remarks

This monograph is one of challenging to "Three Dimension Problems". For the case of $(k_1 = \infty, k_2, k_3 \geq 1)$, the LSTs $W_n^*(s), n = 1, 2, 3$ have been obtained by using the top-down procedure as $\Phi_n(x, y, z) \to P_n(x, y, z) \to \Pi_n(\cdot) \to W_n^*(s)$. It may be possible to extend the dimension-number to ($N = 4$) of the priority model with ($k_1, k_2, k_3, k_4 \geq 1$), however, numerical examples for $N \geq 3$ should be presented as in Ozawa [11] in the future. In the analysis of the exhaustive service priority model with ($k_1 = k_2 = k_3 = \infty$), a new problem is proposed in Remark 4.3, for which the solution has been given in Appendix II. Finally, the classical method used in Katayama and Kobayashi [12] and Katayama [9, 10, 13] has been applied to the multi-dimension problem ($N \geq 3$).

On the other hand, for the shortest queue problem and the dual priority model for the longest queue-length studied by Cohen [6], it seems to be almost impossible to extend to ($N = 3$). The barrier between N=2 and N=3 is very high. We have not yet any mathematical tool of analyzing for these queueing models, such as the boundary value method.

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Appendix I

For the GFs of joint queue-length distributions (not the marginal queue-length distribution), this gives the derivation of $\Pi_n(z_1, z_2, \cdots, z_N)$, which is similar to one of $\Pi_n(z)$ in Eq. (52).

$$\Pi_n(z_1, z_2, \cdots, z_N) = Q_n(z_1, z_2, \cdots, z_N)/\{z_n - Q_n(z_1, z_2, \cdots, z_N)\} \times \{\varphi_n(z_n, z_{n+1}, \cdots, z_N) - \varphi_n(f_n(z_n, z_{n+1}, \cdots, z_N), z_{n+1}, z_{n+2} \cdots, z_N)\}. \quad \text{(A1)}$$

The result is given as the solution of the following functional equation,

$$\varphi_k(z_k, z_{k+1}, \cdots, z_N) - \varphi_k\{f_k(z_k, z_{k+1}, \cdots, z_N), z_{k+1}, \cdots, z_N\} = g_k(z_k, z_{k+1}, \cdots, z_N), \quad \text{(A2)}$$

where $k = n, n+1, \cdots, N$ and f_k, g_k are known functions, see Remark 4.3 in Katayama [13]. First, let us introduce a sequence $\{z_k^{(i)}, z_{k+1}^{(i)}, \cdots, z_N^{(i)}\}$ defined by

$$\{z_k^{(i+1)} := f_k(z_k^{(i)}, z_{k+1}^{(i)}, \cdots, z_N^{(i)}), z_{k+1}^{(i+1)} := z_{k+1}^{(i)}, \cdots, z_N^{(i+1)} := z_N^{(i)}\},\ i = 0, 1, 2, \cdots.$$

Then, by using this relation repeatedly, we have

$$\varphi_k(z_k, z_{k+1}, \cdots, z_N) - \varphi_k(\eta, z_{k+1}, \cdots, z_N) = G_k(z_k, z_{k+1}, \cdots, z_N), (\eta := \lim_{i\to\infty} z_k^{(i)}),$$

where $G_k\{z_k, z_{k+1}, \cdots, z_N | z_k^{(0)} = z_k, z_{k+1}^{(0)} = z_{k+1}, \cdots, z_N^{(0)} = z_N\}$. Therefore, it follows that

$$\varphi_k(z_k, z_{k+1}, \cdots, z_N) = G_k(z_k, z_{k+1}, \cdots, z_N) - G_k(0, z_{k+1}, \cdots, z_N), \quad \text{(A3)}$$

where the boundary condition $\varphi_k(0, z_{k+1}, \cdots, z_N) = 0$ has been used and

$$f_k(z_k, z_{k+1}, \cdots, z_N) := \lambda_{k-1}^+ - x_k(z_k, x_{k+1}, \cdots, z_N) + \sum_{j=k}^{N} \lambda_j(1 - z_j), \quad \text{(A4)}$$

where two terms $x_k(z)$ and $\lambda_k(1 - z_k)$ in Eq. (49a) can be replaced by $x_k(z_k, z_{k+1}, \cdots, z_N)$ and $\sum_{j=k}^{N} \lambda_j(1 - z_j)$, respectively, which provide the explicit expression for $z_k(z_k, z_{k+1}, \cdots, z_N)$, (see Lemma 1 in [13]). That is, f_k is the known function.

$$G_k(z_k, z_{k+1}, \cdots, z_N) := \sum_{i=0}^{\infty} g_k(z_k^{(i)}, z_{k+1}^{(i)}, \cdots, z_N^{(i)} | z_k^{(0)} = z_k, \cdots, z_N^{(0)} = z_N), \quad \text{(A5)}$$

$$g_k(z_k, , z_{k+1}, \cdots, z_N) := \pi_0\{V^*(f_k(z_k, \cdot\cdot, z_N)) - 1\} + \sum_{j=k+1}^{N} \{\varphi_j(z_j, \cdots, z_N) - \varphi_j(f_j, z_{j+1}, \cdots, z_N)\}. \quad \text{(A6)}$$

Therefore, Lemma 4.1 leads to (A1), by using the known $G_{n+1}, G_{n+2}, \cdots, G_N$ and starting from $k = N$. The descent-order procedure is necessary for the derivation of G_n and φ_n.

Appendix II

This is an answer for Remark 4.3. From the exhaustive service at $S_n, n = 1, 2, 3$, we have

$$\Phi_1(0, y, z) = P_1(G_1(y, z), y, z),$$
$$\Phi_2(x, 0, z) = P_2(0, G_2(x, z), z),$$
$$\Phi_3(x, y, 0) = P_3(0, 0, G_3(x, y)).$$

From (24a)∼(24c) and $\sum_{n=1}^{3} \Phi_n(x, y, z) = \Phi_1(0, y, z) + \Phi_2(x, 0, z) + \Phi_3(x, y, 0)$, it follows that

$$\sum_{n=1}^{3} P_n(\cdot) - \sum_{n=1}^{3} \Phi_n(x, y, z) = \phi_0\{V(x, y, z) - 1\}. \quad \text{(B1)}$$

This corresponds to the invariant relation in Lemma 3.1 (Lemma 4.1) using $P_n(\cdot)$ and $\Phi_n(x, y, z)$.

$$\therefore\ \{P_1(x, y, z) - P_1(G_1, y, z)\} + \{P_2(0, y, z) - P_2(0, G_2, z)\} + \{P_3(0, 0, z) - P_3(0, 0, G_3)\} = \phi_0\{V(x, y, z) - 1\}. \quad \text{(B2)}$$

Setting $x = G_1(y,z)$ and $y = G_2(x,z)$ in both-sides of (B2), we obtain

$$P_3(0,0,z) - P_3(0,0,G_3(G_1,G_2)) = \phi_0\{V(G_1(y,z),G_2(x,z),z) - 1\},$$

$$\therefore\ \ \varphi_3(z) - \varphi_3(f_3(z)) = \phi_0 g_3(z), \tag{B3}$$

where

$$\varphi_3(z) := P_3(0,0,z),$$
$$f_3(z) := G_3[G_1\{G_2(\delta_3(z),z),z\},G_2(\delta_3(z),z)],$$
$$g_3(z) := V[G_1\{G_2(\delta_3(z),z),z\},G_2(\delta_3(z),z),z] - 1,$$

and $\delta_3(z) =: x$ is the zero of $x = G_1(y,z) = G_1(G_2(x,z),z)$. Since (B3) is the same type with (16d), we have the solution $\varphi_3(z) = \phi_0\hat{\varphi}_3(z)$.

In succession, setting $x = G_1(y,z)$ and using $\hat{\varphi}_3(z)$, it follows from (B2) that

$$P_2(0,y,z) - P_2(0,G_2(G_1,z),z) = \phi_0\{\hat{\varphi}_3(G_3(G_1,y)) - \hat{\varphi}_3(z) + V(G_1,y,z) - 1\}.$$

$$\therefore\ \ \varphi_2(y,z) - \varphi_2(f_2(y,z),z) = \phi_0 g_2(y,z), \tag{B4}$$

where $\varphi_2(y,z) := P_2(0,y,z)$,

$f_2(y,z) := G_2(G_1(y,z),z)$, $g_2(y,z) := \hat{\varphi}_3(G_3(G_1,y)) - \hat{\varphi}_3(z) + V(G_1,y,z) - 1.$

Let us introduce a sequence $\{y_i, z_i\}$ defined by

$$\{y_{i+1} = f_2(y_i,z_i), z_{i+1} = z_i\}, i = 0,1,2,\cdots.$$

Then, it follows from (B4) that

$$\varphi_2(y_i,z_i) - \varphi_2(y_{i+1},z_{i+1}) = \phi_0 g_2(y_i,z_i).$$

By using this relation repeatedly, we have

$$\varphi_2(y,z) - \varphi_2(\eta,z) = \phi_0\sum_{i=0}^{\infty} g_2(y_i,z_i|y_0 = y, z_0 = z),$$

$$\therefore\ \ \varphi_2(y,z) = \phi_0\{\Gamma_2(y,z) - \Gamma_2(0,z)\}, \tag{B5}$$

where $\Gamma_2(y,z) := \sum_{i=0}^{\infty} g_2(y_i,z_i|y_0 = y, z_0 = z)$ and $\varphi_2(0,z) = 0$ has been used.

Similarly, we have

$$\varphi_1(x,y,z) - \varphi_1(f_1(y,z),y,z) = \phi_0 g_1(x,y,z), \tag{B6}$$

where

$$\varphi_1(x,y,z) := P_1(x,y,z),$$
$$f_1(y,z) := G_1(y,z),$$
$$g_1(x,y,z) := \hat{\varphi}_3(G_3) - \hat{\varphi}_3(z) + \hat{\varphi}_2(G_2,z) - \hat{\varphi}_2(y,z) + V(x,y,z) - 1.$$

Using a sequence $\{x_{i+1} = f_1(y_i,z_i), y_{i+1} = y_i, z_{i+1} = z_i\}, i = 0,1,2,\cdots$, it follows that

$$\therefore\ \ \varphi_1(x,y,z) = \phi_0\{\Gamma_1(x,y,z) - \Gamma_1(0,y,z)\}, \tag{B7}$$

where $\Gamma_1(x,y,z) := \sum_{i=0}^{\infty} g_1(x_i,y_i,z_i|x_0 = x, y_0 = y, z_0 = z)$. ($\varphi_1(0,y,z) = 0$ has been used).

Note that the functional equations (B4) and (B6) are the same type with (A2) in Appendix I. Finally, the descent-order procedure is also necessary for the derivation of $\varphi_n(\cdot)$ and $P_n(\cdot), n = 1,2,3$.

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待ち行列システムの解析における３次元問題

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概要
本稿は，3 つの優先権クラス $1, 2, 3$ を持ち，これらの特性をパラメタ $k_1, k_2, k_3 (k_1, k_2, k_3 > 0)$ によって制御可能な待ち行列システムを解析する．本システムは次のように動作する．システムには 1 つの処理窓口があって，クラス $n (n = 1, 2, 3)$ のキュー内のメッセージを，同キュー内のメッセージ数が，最新のメッセージ到着時点の値よりも k_n だけ減少するまで処理する．その処理が終了したら，最も優先度の高いクラスのキューのメッセージを，同様にメッセージ数がパラメタで指定された値だけ減少するまで処理する．本システムについて，$M_1, M_2, M_3/G_1, G_2, G_3/1$ 優先権付待ち行列システムの確率母関数に基づく解析手法を利用し，各クラスのメッセージにおける待ち時間分布のラプラス・スティルチェス変換を導く．さらに，クラス数が 4 以上の場合への理論の一般化についても述べる．

キーワード
待ち行列理論; 関数方程式; 確率母関数